SOME RADII OF STARLIKENESS AND CONVEXITY PROBLEMS

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CERTIFICATE

This is to certify that the work embodied in the thesis "Some radii of starlikeness and convexity problems" by P.L. Bajpai has been carried out under my supervision and has not been submitted elsewhere for a degree or diploma.

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CONTENTS

CHAPTER 1

INTRODUCTION

1.1 A function $f(z)$ is said to be univalent (or sehlicht) in a domain D, if for any two points z_1 and z_2 of D we have $f(z_1) = f(z_2)$ only if $z_1 = z_2$. A function which is regular and univalent in the unit disc $D^* = \{z: |z| < 1\}$ may be normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The normalization of a function is not an essential restriction, for, if $f(z)$ is univalent, so is the function $g(z) = \frac{f(z) - f(0)}{f'(0)}$. We shall denote by S the class of all analytic functions $f(z)$ which are regular in D and which are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The Taylor expansion of such a function about the origin has the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \tag{1.1.1}$$

The origin of the theory of univalent functions can be traced to a paper by P. Koebe in 1907 on the uniformization of algebraic curves [17]. In this paper Koebe proved in particular that there is a constant K (called koebe constant) such that the boundary of the map of $|z| < 1$ by any function $f(z) \in S$ is always at a distance not less than K from $f(z) = 0$. Koebe's work was followed by a number of eminent mathematicians (Gronwall [12] , Faber [9] , Bieherhach [5] and others.) In particular, Bieberhach in 1916 obtained that $K = 1/4$, which could also be found from the results of Gronwall [12]. Bieberhach also found that $|a_2| \leq 2$ for $f(z) \in S$. Since the equality in above result is attained

* Hereinafter we shall denote the unit disc $|z| < 1$ by D

for the functions (Koebe functions)

$$(1.1.2) \qquad f(z) = z\,(1 + e^{i\theta} z)^{-2} \quad ,$$

θ real, and also $|a_n| = n$, $n = 2,3,\ldots$ for the above functions. It was conjectured by Bieherhach that

$$(1.1.3) \qquad |a_n| \leq n, \; n = 2,3,\ldots.$$

for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ and equality is attained only for Koebe functions. Uptil now conjecture has been settled for $n = 2,3,4,5$ and 6 by Bieberhach [5] , Löwner [23] , Garabedian and Schiffer [10] R.N. Pederson [28] and R.N. Rederson and Schiffer [29] . But in general the inequality (1.1.3) still poses an open problem.

Failure of establishing the Bieberbach conjecture gives rise the investigation of various subclasses of S. First important subclass S is the class of convex functions.

A region E is said to be convex if the line joining any two points of E lies wholly in E. A function $f(z) \in S$ is said to be convex in D if $f(z)$ maps D onto a convex domain. Denote by $C(\alpha)$ the class of all functions $f(z)$ which satisfy the condition

$$(1.1.4) \qquad \mathrm{Re}\; \{1 + \frac{zf''(z)}{f'(z)}\} > \alpha \quad \text{for} \quad z \in D.$$

Functions in the class $C(\alpha)$ are known as convex functions of order α , $0 \leq \alpha < 1$.

A region E containing the point $w = 0$ is said to be starlike with respect to point $w=0$ if the line segment joining $w=0$ to any other point of E lies completely in E. A function $f(z) \in S$ is said to be

starlike in D if $f(z)$ maps D onto a starlike region. Denote by $S^*(\alpha)$ the class of functions $f(z) \in S$ which satisfy the condition

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in D. \tag{1.1.5}$$

Functions in the class $S^*(\alpha)$ are known as starlike functions of order α, $0 \leq \alpha < 1$.

It follows immediately that a necessary and sufficient condition for $f(z) \in S$ to map D onto a convex domain is that $zf'(z)$ map D onto a starlike domain with respect to origin.

A class wider than the class of starlike functions is the class of Spiral-like functions introduced by L. Špaček [35] in 1932. Špaček essentially showed that a function $f(z) \in S$ is spiral like in D if

$$\text{Re}\left\{\frac{\xi zf'(z)}{f(z)}\right\} > 0, \quad |\xi| = 1, \; z \in D. \tag{1.1.6}$$

If we replace ξ by $e^{i\alpha}$, $|\alpha| \leq \pi/2$, then f is called univalent α-spiral function introduced by R.J. Libera [20]. Clearly starlike functions are also α-spiral functions with $\alpha = 0$. We denote class of α-spiral functions by $S(\alpha)$.

Another class wider than that of star-like functions is the class of close-to-convex functions, which we shall denote by K. This class of close-to-convex functions has been introduced by W. Kaplan [16]. If $f(z)$ is analytic in D, then $f(z)$ is close-to-convex if there exists a function $\phi(z) \in C$ such that

$$\text{Re}\left\{\frac{f'(z)}{\phi'(z)}\right\} > 0 \quad \text{for } z \in D. \tag{1.1.7}$$

This class of functions includes the class S^*. Kaplan [16] also characterized close-to-convex functions, without reference to a convex function. Thus, $f(z)$ is close-to-convex if and only if

$$\int_{\theta_1}^{\theta_2} \text{Re}\,\{1 + \frac{zf''(z)}{f'(z)}\}\, d\theta > -\pi \tag{1.1.8}$$

where $\theta_1 < \theta_2$, $z = re^{i\theta}$ and $r < 1$.

Some analogous extensions ([15] and [19]) of classed $C(\alpha)$, $S^*(\alpha)$ and K are also carried over to the meromorphic univalent functions which are regular in the unit disc except at a point $z = 0$.

Let $M(\alpha)$ be the class of all meromorphic univalent functions

$$f(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + \ldots \,. \tag{1.1.9}$$

which satisfy the condition

$$-\text{Re}\,\{1 + \frac{zf''(z)}{f'(z)}\} > \alpha \text{ for } z \in D. \tag{1.1.10}$$

Denote by $\Sigma^*(\alpha)$ the class of functions of the form (1.1.9) which satisfy the condition

$$-\text{Re}\,\{\frac{zf'(z)}{f(z)}\} > \alpha\ , \ z \in D. \tag{1.1.11}$$

Functions in the classes $M(\alpha)$ and $\Sigma^*(\alpha)$ are known as meromorphic convex and starlike functions of order α , $0 \leq \alpha < 1$, respectively.

1.2 Out lines of the thesis.

Usually the following types of problems are studied for the univalent functions:

(a) Distortion theorems, i.e., determination of upper and lower estimates for

$$|f(z)| \;,\quad |f'(z)| \text{ and } \left|\frac{zf'(z)}{f(z)}\right| \;.$$

(b) Coefficient estimates.

(c) Bounds for $\arg \{\frac{f(z)}{z}\}$ and $\arg \{f'(z)\}$.

(d) Radii of univalency, starlikeness and convexity.

In the present work, we have mainly restricted our investigations to the problems of type (d) for certain subclasses of univalent functions and meromorphic univalent functions.

In a recent paper R.J. Libera and A.E. Livingston [21] determined the disc in which

$$(1.2.1) \qquad f(z) = \frac{1}{2} \; [z\, F(z)]'$$

(where $F(z) \in S^*(\alpha)$) is starlike of order β if, $0 \le \alpha \le \beta < 1$. They were unable to obtain suitable results for the complementary case, i.e., when $\beta < \alpha$. In the 3rd section of the chapter two we give a method which covers both the cases. The technique used here are the modified forms of the techniques used by V. Singh and R.M. Goel [34] We have also determined the disc in which $f(z) \in S^*(\alpha)$ is convex of order β , $0 \le \beta < 1$. In the 5th and 6th sections of this chapter we have determined the radii of convexity and starlikeness for some classes of regular functions in D . In the last section of this chapter the radius of convexity of the class defined by D.J. Wright [37] has been determined. In particular the results of this chapter include the results of R.J. Libera and A.E. Livingston [21] , V.A. Zmorovič [39] ,

V. Singh and R.M. Goel [34] , J.S. Rathi [30] , D.J. Wright [37] and Ram Singh [32] .

In the 3rd section of the chapter three the radius of convexity for the class $\Sigma^*(\alpha)$ is determined. Some radius of starlikeness problems for the class of meromorphic functions have also been studied in this chapter. In the special cases the results of V.A. Zmorovič [39] and K.S. Padmanabhan [27] follow from the results of this chapter.

3rd section of the chapter four deals with the determination of the effect of second coefficient on the radius of starlikeness of $f(z) = p(z) - 1 = 2az + a_2z^2+\dots$ where $p(z)$ is regular, $\mathrm{Re}\ \{p(z)\} > \alpha$, $0 \leq \alpha < 1$, in D and $|a| < 1 - \alpha$. In the ivth and vth sections of this chapter the effect of second coefficient on the radii of convexity and starlikeness of the functions $f(z) \varepsilon S^*$ and $f(z)$ of the form (1.2.1) has been investigated respectively. In particular results of this chapter include the results of R.S. Gupta [13] and A.E. Livingston [22] .

In the last chapter of this work the effect of dropping first (n-1) coeficients of Taylor expansion of a function about origin on the radii of convexity and starlikeness of certain classes of analytic functions is determined. In particular results of this chapter include the results of S.K. Bajpai & R.S.L. Srivastava [3], Causey and Merkas [7], J.S. Ratti [31],[30], D.J. Wright [37] and Ram Singh [32] .

CHAPTER 2

THE RADII OF CONVEXITY AND STARLIKENESS

2.1 Let S denote the class of regular and univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $D = \{z: |z| < 1\}$. For fixed α, $0 \leq \alpha < 1$, let $S^*(\alpha)$ denote the subclass of S consisting of functions $f(z)$ satisfying the condition

$$(2.1.1) \qquad \operatorname{Re} \left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \text{ for } z \in D.$$

For fixed α, $0 \leq \alpha < 1$, let $C(\alpha)$ denote the subclass of S consisting of functions $f(z)$ satisfying the condition

$$(2.1.2) \qquad \operatorname{Re} \left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \text{ for } z \in D.$$

Functions in the classes $S^*(\alpha)$ and $C(\alpha)$ are known as starlike and convex functions of order α respectively.

V_α denotes the class of functions

$$(2.1.3) \qquad g(z) = \frac{1}{2}\,(F(z) + zF'(z))$$

where $F(z) \in S^*(\alpha)$.

Let $P(\alpha)$ be the class of functions

$$(2.1.4) \qquad p(z) = 1 + c_1 z + c_2 z^2 + \ldots$$

which are regular in D and have real part greater than α, $0 \leq \alpha < 1$, in D

M stands for the class of functions

(2.1.5) $$m(z) = \frac{z}{2g(z)}\,[F(z) + zF'(z)]$$

where $g(z) \in C$, $S^*(\alpha)$ or $\frac{g(z)}{z} \in P(\beta)$ and $F(z) \in S^*(\gamma)$.

Let R represents the class of functions $r(z)$ such that

(2.1.6) $$r'(z) = f'(z)\,h(z)$$

where $h(z) \in P(\alpha)$ and $\mathrm{Re}\, f'(z) > 0$, $f(z) \in S^*(\alpha)$ or $f(z) \in C(\alpha)$. $f'(z)$ and $r'(z)$ stand for the derivatives of $f(z)$ and $r(z)$ respectively. Let $H(\alpha)$ be the class of functions

(2.1.7) $$F(z) = z + a_2 z^2 + \dots$$

satisfying the condition

(2.1.8) $$\left|\frac{zF'(z)}{F(z)} - 1\right| < \alpha, \quad (0 < \alpha \leq 1) \text{ for } z \in D.$$

In this chapter the radii of convexity of classes $S^*(\alpha)$, R and $H(\alpha)$ and radii of starlikeness of classes V_α and M are determined.

2.2 <u>We start with proving the following lemmas.</u>

<u>Lemma</u> 1 : <u>Let</u>

(2.2.1) $$H(z) = \frac{a}{1+z\phi(z)} - \frac{1}{1+bz\phi(z)} - \frac{(1-b)z^2\phi'(z)}{(1+z\phi(z))(1+bz\phi(z))}$$

<u>where</u> $\phi(z)$ <u>is regular and</u> $|\phi(z)| \leq 1$ <u>in</u> D, $-1 < b \leq 1$ <u>and</u> $a \geq 1$. <u>Then for</u> $|z| = r$, $0 \leq r < 1$,

(2.2.2) $$\mathrm{Re}\,\{H(z)\} \leq \frac{a-1 + (1 - ab)\,r}{(1 - r)\,(1 - br)}$$

<u>and</u>

$$(2.2.3)\quad \mathrm{Re}\{H(z)\} \geq \begin{cases} -\dfrac{(1+ab+2b)(1-r^2)+2(1-b)}{(1-b)\,(1-r^2)} + \dfrac{2}{1-b}\sqrt{\dfrac{(1+a)(1+b)(1-br^2)}{1-r^2}} \\ \quad \underline{\text{for}} \quad u_o \geq u_1 \\ \quad \underline{\text{and}} \\ \dfrac{a-1+(ab-1)r}{(1+r)\,(1+br)} \; \underline{\text{for}} \quad u_o \leq u_1 \end{cases}$$

where $u_o = \frac{1}{1-b}\left\{\sqrt{\frac{(1+b)\,(1-br^2)}{(1+a)\,(1-r^2)}} - b\right\}$ and $u_1 = \frac{1}{1+r}$

Proof : From (2.2.1) we have

$$(2.2.4)\qquad \mathrm{Re}\,\{H(z)\} \leq \mathrm{Re}\,\left\{\frac{a}{1+z\phi(z)}\right\} - \mathrm{Re}\,\left\{\frac{1}{1+bz\phi(z)}\right\}$$

$$+\frac{(1-b)(|z|^2 - |z\phi(z)|^2)}{(1-|z|^2)\ (1+z\phi(z)|\ \ |(1+bz\phi(z)|}.$$

The above inequality has been obtained by using the well-known result [25, 168]

$$(2.2.5)\qquad |\phi'(z)| \leq \frac{1-|\phi(z)|^2}{1-|z|^2}$$

Let $|z| = r$ and $\frac{1}{1+z\phi(z)} = u\ e^{iv}$, then u satisfies

$$(2.2.6)\qquad \left|u - \frac{1}{1-|z|^2}\right| \leq \frac{|z|}{1-|z|^2}$$

From (2.2.4) we have

$$(2.2.7)\qquad \mathrm{Re}\,\{H(z)\} \leq [a\,u\cos v - \frac{u^2(1-b)+bu\cos v}{u^2(1-b)^2+2b(1-b)u\cos v+b^2}$$

$$+\frac{(1-b)\,(r^2u^2-1+2u\cos v-u^2)}{(1-r^2)\sqrt{\{u^2(1-b)^2+2b(1-b)u\cos v+b^2\}}}]$$

$$\equiv G(u,v,r)\quad (\text{say.})$$

Diff. G partially with respect to v we obtain

$$(2.2.8)\quad \frac{\partial G}{\partial v} = -u \sin v \,[a - \frac{b(b^2-(1-b)^2u^2)}{\{u^2(1-b)^2+2b(1-b)u\cos v+b^2\}^2}$$

$$+ \frac{2\,(1-b)}{(1-r^2)\{u^2(1-b)^2+2b(1-b)u\cos v+b^2\}^{1/2}}$$

$$- \frac{b(1-b)^2(r^2u^2-1+2u\cos v-u^2)}{(1-r^2)\{u^2(1-b)^2+2b(1-b)u\cos v+b^2\}^{3/2}}\,]$$

Since $|z\phi(z)| \leq r$ for $|z| = r$ and $|z\phi(z)|^2 = \dfrac{1-2u\cos v+u^2}{u^2}$, we can easily conclude that

$$(2.2.9)\qquad r^2u^2 - 1 + 2u\cos v - u^2 \geq 0$$

and from (2.2.6)

$$\frac{1}{1+r} \leq u \leq \frac{1}{1-r}\,.$$

The quantity within square brackets of (2.2.8) is positive for positive values of b. For negative b the minimum of the quantity within square brackets of (2.2.8) is attained at one of the end points of the interval $[\frac{1}{1+r}, \frac{1}{1-r}]$ and also values at these points are positive. Hence maximum of G occurs at $v = 0$. Putting $v = 0$ in (2.2.7) we have

$$(2.2.11)\qquad g(u,r) \equiv G(u,0,r)$$

$$= \frac{1+r^2}{1-r^2} + u(a-1) - \frac{1+br^2}{(1-r^2)\,(b+(1-b)u)}\,.$$

The above expression for $g(u,r)$ allows us to claim that it is an monotone increasing function of u, therefore its maximum occurs at

$$(2.2.12)\qquad u_2 = \frac{1}{1-r}$$

and is equal to

$$(2.2.13) \qquad g(u_2,r) = \frac{a-1+(1-ab)r}{(1-r)(1-br)}$$

Thus the proof of the first part is complete.

Now using (2.2.5) in (2.2.1) and by taking $|z| = r$ and $\frac{1}{1+z\phi(z)} = u + iv$ we have

$$(2.2.14) \quad \operatorname{Re}\{H(z)\} \geq \left[au - \frac{bu+(1-b)u^2+(1-b)v^2}{(b+(1-b)u)^2+(1-b)^2v^2} - \frac{(1-b)\{r^2(u^2+v^2)-(1-u)^2-v^2\}}{(1-r^2)\sqrt{(b+(1-b)u)^2+(1-b)^2v^2}}\right] \equiv L(u,v,r) \quad \text{(say.)}$$

Diff. L w.r.t. v partially we obtain

$$(2.2.15) \quad \frac{\partial L}{\partial v} = v\left[\frac{-2b(1-b)(b+(1-b)u)}{\{(b+(1-b)u)^2+(1-b)^2v^2\}^2} + \frac{2(1-b)}{\sqrt{(b+(1-b)u)^2+(1-b)^2v^2}} + \frac{(1-b)^3(r^2(u^2+v^2)-(1-u)^2-v^2)}{(1-r^2)\{(b+(1-b)u)^2+(1-b)^2v^2\}^{3/2}}\right]$$

Minimum of L occurs at $v = 0$ because quantity within square brackets of (2.2.15) is strictly positive in view of (2.2.9) and (2.2.10). On putting $v = 0$ in (2.2.14) we obtain

$$\ell(u,r) \equiv L(u,0,r) = au - \frac{u}{b+(1-b)u} - \frac{(1-b)(r^2u^2-(1-u)^2)}{(1-r^2)(b+(1-b)u)}$$

$$= -\left(\frac{1+b}{1-b} + \frac{2}{1-r^2}\right) + (a+1)u + \frac{(1+b)(1-br^2)}{(1-r^2)(1-b)(b+(1-b)u)}$$

Absolute minimum of ℓ in $(0, \infty)$ is attained at

$$b + (1-b)u_0 = \sqrt{\frac{(1+b)(1-br^2)}{(1+a)(1-r^2)}}$$

i.e.,

(2.2.16) $$u_o = \frac{1}{1-b}\left\{\sqrt{\frac{(1+b)(1-br^2)}{(1+a)(1-r^2)}} - b\right\}$$

provided $u_o \in [\frac{1}{1+r}, \frac{1}{1-r}]$ and is equal to

(2.2.17) $$\ell(u_o,r) \equiv -\frac{(1+ab+2b)(1-r^2)+2(1-b)}{(1-b)(1-r^2)} + \frac{2}{1-b}\sqrt{\frac{(1+a)(1+b)(1-br^2))}{1-r^2}}$$

It is easy to check that $u_o < \frac{1}{1-r}$ but u_o is not always greater than $\frac{1}{1+r}$. In such a case when $u_o \notin [\frac{1}{1+r}, \frac{1}{1-r}]$ the minimum of $\ell(u,r)$ on the segment $[\frac{1}{1+r}, \frac{1}{1-r}]$ is attained at

(2.2.18) $$u_1 = \frac{1}{1+r}$$

and is equal to

(2.2.19) $$\ell(u_1,r) \equiv \frac{a-1+(ab-1)r}{(1+r)(1+br)}$$

(2.2.20) $$\ell(u_o,r) = \ell(u_1,r)$$

for such values of a and b for which

(2.2.21) $$u_o = u_1$$

<u>Lemma</u> 2: <u>If</u> $p(z) \in P(\alpha)$ <u>and</u> $|z| \leq r < 1$, <u>then</u>

(2.2.22) $$\mathrm{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \leq \frac{2r(1-\alpha)}{(1-r)(1-(2\alpha-1)r)}$$

<u>with equality only for</u>

(2.2.23) $$p(z) = \frac{1+(1-2\alpha)\varepsilon z}{1-\varepsilon z}, \quad |\varepsilon| = 1.$$

<u>Proof</u> : Since $p(z) \in P(\alpha)$ there exists a function $w(z) = z\phi(z)$ satisfying Schwarz's lemma such that

$$(2.2.24) \qquad p(z) = \frac{1+(2\alpha-1)\,z\phi(z)}{1+z\phi(z)}$$

Logarithmic diff. of (2.2.24) yields

$$(2.2.25) \qquad \frac{zp'(z)}{p(z)} = \frac{1}{1+z\phi(z)} - \frac{1}{1+(2\alpha-1)z\phi(z)} - \frac{2(1-\alpha)\,z^2\phi'(z)}{(1+z\phi(z))(1+(2\alpha-1)z\phi(z))}$$

Taking $a = 1$, $b = 2\alpha - 1$ in (2.2.2), we obtain (2.2.22).

The above lemma has also been proved by R.J. Libera [18] by using different technique.

Lemma 3. Let α satisfy $0 \leq \alpha < 1$. Let $r(\alpha)$ denote the root unique in $(2-\sqrt{3}, 1]$ of the equation

$$(2.2.26) \qquad (1-2\alpha)\,r^3 - 3\,(1-2\alpha)\,r^2 + 3r - 1 = 0.$$

If $p(z) \in P(\alpha)$, then for $|z| \leq r < 1$

$$(2.2.27) \quad Re\left\{\frac{zp'(z)}{p(z)}\right\} \geq \begin{cases} \dfrac{-2r(1-\alpha)}{(1+r)(1+(2\alpha-1)r)} & \text{for } 0 \leq r \leq r(\alpha) \\ \text{and} & \\ -\dfrac{(\sqrt{1+(1-2\alpha)r^2} - \sqrt{\alpha(1-r^2)})^2}{(1-\alpha)\,(1-r^2)} & \text{for } r(\alpha) \leq r < 1. \end{cases}$$

Equality sign in the first inequality of (2.2.27) occurs for the function given by (2.2.23) for $0 \leq r \leq r(\alpha)$ and that in the second inequality for the function

$$(2.2.28) \qquad p(z) = \frac{1-2\alpha\lambda\varepsilon z + (2\alpha-1)\varepsilon^2 z^2}{1-2\varepsilon\lambda z + \varepsilon^2 z^2}, \quad |\varepsilon| = 1$$

for $r(\alpha) \leq r < 1$, where

$$\lambda = \frac{2r}{1-r^2} - \frac{(1-r^2)^{3/2}}{2r(1+r^2)} \left(\frac{1+(1-2\alpha)r^2}{\alpha}\right)^{1/2}, \quad \alpha > 0$$

Proof : From (2.2.25)

$$\frac{zp'(z)}{p(z)} = \frac{1}{1+z\phi(z)} - \frac{1}{1+(2\alpha-1)z\phi(z)} - \frac{2(1-\alpha)z^2\phi'(z)}{(1+z\phi(z))(1+(2\alpha-1)z\phi(z))}$$

Taking $a = 1$, $b = 2\alpha-1$ in (2.2.3)

$$(2.2.29) \quad \mathrm{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geq \begin{cases} \dfrac{-2r(1-\alpha)}{(1+r)\,(1+(2\alpha-1)r)}, & \text{for } u_o \leq u_1 \\ -\dfrac{(\sqrt{1+(2\alpha-1)r^2} - \sqrt{\alpha(1-r^2)})^2}{(1-\alpha)\,(1-r^2)}, & \text{for } u_o \geq u_1 \end{cases}$$

where $u_o = \dfrac{1}{2(1-\alpha)} \left\{\sqrt{\dfrac{\alpha(1-(2\alpha-1)r^2)}{(1-r^2)}} - (2\alpha-1)\right\}$ and $u_1 = \dfrac{1}{1+r}$.

The first and second inequalities of (2.2.29) becomes equal for such values of α for which

$$u_o = u_1$$

i.e.,

$$\frac{\alpha(1-(2\alpha-1)r^2)}{1-r} = \frac{(1+(2\alpha-1)r)^2}{1+r}$$

or

$$g_1(\alpha,r) \equiv (1-2\alpha)r^3 - 3(1-2\alpha)r^2 + 3r - 1 = 0$$

$g_1(\alpha,r)$ is a strictly increasing function of r, $0 \leq r < 1$, for each α, $0 \leq \alpha < 1$.

$g_1(\alpha,(2-\sqrt{3})) = 2(1-\alpha)\,(5-3\sqrt{3}) < 0$

$g_1(\alpha,1) = 4\alpha \geq 0$.

Thus $g_1(\alpha,r)$ has unique root $r(\alpha)$ in $(2-\sqrt{3},1]$.

Hence proof of the lemma is complete.

The above lemma has also been proved by V.A. Zmorovič [40] by using different techniques.

Lemma 4. Let $p(z) \in P(\alpha)$ and $|z| \leq r < 1$, then

$$(2.2.30) \qquad \operatorname{Re}\{p(z)\} \geq \frac{1+(2\alpha-1)r}{1+r}$$

equality occurs in (2.2.30) only for functions of the form (2.2.23).

Proof of the above lemma is simple and hence omitted.

Lemma 5. Let α satisfy $0 \leq \alpha < 1$. Let $r_1(\alpha)$ denote the root unique in $(\frac{1}{2}, 1]$ of the equation

$$(2.2.31) \qquad \alpha(2\alpha-1)\, r^3 + \alpha(7-2\alpha)\, r^2 + (5-4\alpha)\, r - 3 = 0.$$

If $g(z) \in V_\alpha$, then for $|z| = r$, $0 \leq r < 1$,

$$(2.2.32) \qquad \operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \leq \frac{1 + (2-4\alpha)\, r+\alpha(2\alpha-1)\, r^2}{(1-r)\,(1-\alpha r)},$$

$$(2.2.33) \operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geq \begin{cases} \frac{2}{1-\alpha}\{\sqrt{(1+\alpha)(4-2\alpha)a} -1-a\} \\ \quad \text{for } r_1(\alpha) \leq r < 1, \\ \frac{1-(2-4\alpha)r+\alpha(2\alpha-1)r^2}{(1+r)\,(1+\alpha r)} \\ \quad \text{for } 0 \leq r \leq r_1(\alpha), \end{cases}$$

where $a = \frac{1-\alpha r^2}{1-r^2}$. Equality sign in (2.2.32) and the second inequality of (2.2.33) is attained for the function

$$(2.2.34) \qquad F(z) = \frac{z}{(1+z)^{2(1-\alpha)}}$$

for $0 \leq r \leq r_1(\alpha)$

and that in first inequality for the function

(2.2.35) $$F(z) = z\,(1-2bz+z^2)^{-1+\alpha}$$

for $r_1(\alpha) \leq r < 1$

where b is determined from

(2.2.36) $$\frac{1-(1+\alpha)\,br+\alpha r^2}{1-2br+r^2} = \sqrt{\frac{(1+\alpha)\,a}{4-2\alpha}} = R_o$$

Proof: Since $F(z) \in S^*(\alpha)$ there exists a function $w(z) = z\phi(z)$ satisfying Schwarz's lemma such that

(2.2.37) $$\frac{zF'(z)}{F(z)} = \frac{1+(2\alpha-1)\,z\phi(z)}{1+z\phi(z)}, \quad z \in D.$$

Therefore

$$g(z) = F(z)\,\left[\frac{1+\alpha\,z\phi(z)}{1+z\,\phi(z)}\right]$$

Diff. $g(z)$ logarithmically w.r.t. z and then using (2.2.37)

(2.2.38) $$\frac{zg'(z)}{g(z)} = 2\alpha-1+\frac{3-2\alpha}{1+z\phi(z)} - \frac{1}{1+\alpha z\phi(z)} - \frac{(1-\alpha)z^2\phi'(z)}{(1+z\phi(z))(1+\alpha z\phi(z))}$$

Taking $a = 3-2\alpha$ and $b = \alpha$ in (2.2.2) and (2.2.3), we get (2.2.32) and

(2.2.39) $$\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geq \begin{cases} \dfrac{2}{1-\alpha}\left\{\sqrt{(1+\alpha)(4-2\alpha)\,a} - 1 - a\right\}, & \text{for } u_o \geq u_1, \\[2ex] \dfrac{1-(2-4\alpha)r+\alpha(2\alpha-1)\,r^2}{(1+r)\,(1+\alpha r)}, & \text{for } u_o \leq u_1 \end{cases}$$

where $u_o = \frac{1}{1-\alpha}\left\{\sqrt{\frac{(1+\alpha)\,a}{4-2\alpha}} - \alpha\right\}$ and $u_1 = \frac{1}{1+r}$

First and second inequalities of (2.2.39) become equal for such values of α for which

$$u_o = u_1,$$

i.e.,

$$\frac{(1-\alpha r)^2}{1-r} = \frac{(1+\alpha)\,(1-\alpha r^2)}{(4-2\alpha)\,(1+r)}$$

or

$$g_2(\alpha,r) \equiv \alpha(2\alpha-1)\,r^3 + \alpha(7-2\alpha)\,r^2 + (5-4\alpha)\,r - 3 = 0$$

It is easy to check that $g_2\ (\alpha,\ r)$ is a strictly increasing function of r, $0 \leq r \leq 1$, for each α, $0 \leq \alpha < 1$.

$$g_2(\alpha,1/2) = -\frac{\alpha^2}{4} - \frac{3}{8}\alpha - \frac{1}{2} < 0 \quad \text{and}$$

$$g_2(\alpha,1) = 2(1-\alpha) > 0$$

Thus $g_2(\alpha,\ r)$ has an unique root $r_1(\alpha)$ in $(\frac{1}{2},1]$.

Hence proof of the lemma is complete.

2.3 <u>Radius of starlikeness for the class</u> V_α .

<u>Theorem</u> 1 : <u>Let</u> $g(z) \in V_\alpha$ <u>and</u> $r_1(\alpha)$ <u>the root, unique in</u> $(\frac{1}{2}, 1]$ <u>of the equation</u> (2.2.31). <u>Then</u> $g(z)$ <u>is starlike of order</u> β <u>for</u> $|z| < r_o$, <u>where</u> r_o, <u>is the smallest positive root of the equation</u>

$$(2.3.1) \qquad 1-\beta + (2(2\alpha-1) - \beta(1+\alpha))\,r + \alpha(2\alpha-\beta-1)\,r^2 = 0$$

<u>if</u> $0 \leq r_o \leq r_1(\alpha)$,

<u>and of the equation</u>

(2.3.2) $$a^2 + \{\beta(1-\alpha) - 2(1+\alpha-\alpha^2)\} a + \{1 + \frac{\beta(1-\alpha)}{2}\}^2 = 0$$

<u>if</u> $r_1(\alpha) \leq r_o < 1$, <u>where</u> $a = \frac{1-\alpha r^2}{1-r^2}$. <u>This result is sharp.</u>

<u>Proof</u> : Since $g(z) \varepsilon V_\alpha$ from Lemma 5 we get

(2.3.3) $$\text{Re}\,\{\frac{zg'(z)}{g(z)} - \beta\} \geq \frac{1 - (2-4\alpha)\, r + \alpha(2\alpha-1)\, r^2}{(1+r)\,(1+\alpha r)} - \beta$$

$$= \frac{1-\beta + (2(2\alpha-1)-\beta(1+\alpha))r+\alpha(2\alpha-\beta-1)\, r^2}{(1+r)\,(1+\alpha r)}$$

if $0 \leq r \leq r_1(\alpha)$,

and

(2.3.4) $$\text{Re}\,\{\frac{zg'(z)}{g(z)} - \beta\} \geq \frac{2}{1-\alpha}\,\{\sqrt{(1+\alpha)(4-2\alpha)\, a} - 1 - a - \frac{\beta(1-\alpha)}{2}\}$$

if $r_1(\alpha) \leq r < 1$.

Therefore

(2.3.5) $$\text{Re}\,\{\frac{zg'(z)}{g(z)} - \beta\} \geq 0 \text{ if}$$

$$(1-\beta) + (2(2\alpha-1)-\beta(1+\alpha))\, r + \alpha(2\alpha-\beta-1)\, r^2 \geq 0$$

and

(2.3.6) $$a^2 + \{\beta(1-\alpha) - 2\,(1+\alpha-\alpha^2)\}a + \{1 + \frac{\beta(1-\alpha)}{2}\}^2 \geq 0$$

(2.3.5) is valid only when $0 \leq r \leq r_o \leq r_1(\alpha)$ and

(2.3.6) is valid only when $r_1(\alpha) \leq r \leq r_o < 1$.

The equality sign in (2.3.3) is attained for the function given by

(2.2.34) and that in (2.2.4) for the function given by (2.2.35).

By taking $\beta = 0$ in (2.3.1) and (2.3.2) we obtain Theorem 3.2 of V. Singh and R.M. Goel [34] as a corollary of the above Theorem.

One can easily see that the Theorem 1 of Libera and Livingston [21] follows from (3.1) which they obtained under the restriction $\alpha \leq \beta < 1$. Thus if $\beta = 0, \alpha$ has to be zero. In our case if $\beta = 0, \alpha$ need not be zero.

Note : Lemma 5 and Theorem 1 are to appear [1]

2.4 Radius of convexity for the class $S^*(\alpha)$.

Theorem 2: If $f(z) \in S^*(\alpha)$, and $r'(\alpha)$ the root, unique in $(2.\sqrt{3},1]$, of the equation

$$(2.4.1) \qquad -(2\alpha-1)(1-\alpha)r^3 + (2\alpha-1)(3-\alpha)r^2 + (3-2\alpha)r - 1 = 0.$$

Then $f(z)$ is convex of order β for $|z| < r_0'$, where r_0' is the smallest positive root of the equation

$$(2.4.2) \qquad 1-\beta - (4+2\alpha\beta-6\alpha)r + (2\alpha-1)(2c-\beta-1)r^2 = 0$$

if $0 \leq r \leq r'(\alpha)$,

and

$$(2.4.3) \qquad a'^2 + \{2\beta(1-\alpha) - 2\alpha(3-2\alpha)\}a' + \{(\alpha+\beta(1-\alpha)\}^2 = 0$$

if $r'(\alpha) \leq r < 1$,

where $a' = \dfrac{1-(2\alpha-1)r^2}{1-r^2}$. This result is sharp.

Proof : Since $f(z) \varepsilon S^*(\alpha)$, therefore $\frac{zf'(z)}{f(z)} \varepsilon P(\alpha)$.

Hence

$$\frac{zf'(z)}{f(z)} = \frac{1+(2\alpha-1)\, z\phi(z)}{1 + z\phi(z)} \tag{2.4.4}$$

Diff. (2.4.4) logarithmically w.r.t. z and combining it with (2.4.4) we get

$$1 + \frac{zf''(z)}{f'(z)} = 2\alpha-1 + \frac{3-2\alpha}{1+z\phi(z)} - \frac{1}{1+(2\alpha-1)z\phi(z)}$$

$$- \frac{2(1-\alpha)z^2\phi'(z)}{(1+z\phi(z))(1+(2\alpha-1)z\phi(z))}$$

Taking $a = 3-2\alpha$ and $b = 2\alpha -1$ in (2.2.3) we get

$$\mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\} \geq \begin{cases} \frac{1}{1-\alpha}\{2\sqrt{\alpha(2-\alpha)a'} - \alpha - a'\} \\ \text{for } u_o \geq u_1 , \\ \frac{1-(4-6\alpha)r+(2\alpha-1)^2r^2}{(1+r)(1+(2\alpha-1)r)} \\ \text{for } u_o \leq u_1 \end{cases} \tag{2.4.5}$$

where $u_o = \frac{1}{2(1-\alpha)}\{\sqrt{\frac{\alpha a'}{2-\alpha}} - (2\alpha-1)\}$. and $u_1 = \frac{1}{1+r}$.

First and second inequalities of (2.4.5) become equal for such values of α for which

$$u_o = u_1$$

i.e.,

$$\frac{\alpha(1-(2\alpha-1)r^2)}{(2-\alpha)(1-r)} = \frac{(1+(2\alpha-1)r)^2}{1+r}$$

or

$$g_2(r,\alpha) = -(2\alpha-1)(1-\alpha)r^3+(2\alpha-1)(3-\alpha)r^2+(3-2\alpha)r-1 = 0$$

It is easy to verify that $g_2(r,\alpha)$ is strictly increasing function of r, $0 \leq r < 1$, for each α, $0 \leq \alpha < 1$.

$$g_2(2-\sqrt{3},\alpha) = 10-6\sqrt{3} + 19\sqrt{3}\alpha - 33\alpha + 38\alpha^2 - 22\sqrt{3}\alpha^2 < 0$$

$$g_2(1,\alpha) = 2\alpha \geq 0$$

Therefore $g_2(r,\alpha)$ has an unique root $r(\alpha)$ in $(2-\sqrt{3}, 1]$.

Thus (2.4.5) is equivalent to

$$(2.4.6)\quad \operatorname{Re}\{1 + \frac{zf''(z)}{f'(z)}\} \geq \begin{cases} \frac{1}{1-\alpha}\{2\sqrt{\alpha(2-\alpha)\,a'} - \alpha - a'\} \\ \quad \text{for } r'(\alpha) \leq r < 1, \\ \frac{1-(4-6\alpha)r+(2\alpha-1)^2r^2}{(1+r)(1+(2\alpha-1)r)} \\ \quad \text{for } 0 \leq r \leq r'(\alpha). \end{cases}$$

From (2.4.6), we have

$$\operatorname{Re}\{1 + \frac{zf''(z)}{f'(z)} - \beta\} \geq \frac{1 - (4-6\alpha)\,r + (2\alpha-1)^2\,r^2}{(1+r)\ (1+(2\alpha-1)r)} - \beta$$

$$= \frac{1-\beta - (4+2\alpha\beta-6\alpha)r+(2\alpha-1)\ (2\alpha-\beta-1)r^2}{(1+r)\ (1+(2\alpha-1)r)}$$

if $0 \leq r \leq r'(\alpha)$,

and

$$\operatorname{Re}\{1 + \frac{zf''(z)}{f'(z)}\} \geq \frac{1}{1-\alpha}\{2\sqrt{\alpha(2-\alpha)a'} - \alpha - a' - \beta(1-\alpha)\}$$

if $r'(\alpha) \leq r < 1$.

Therefore

$$(2.4.2)\qquad \operatorname{Re}\{1 + \frac{zf''(z)}{f'(z)} - \beta\} \geq 0 \quad \text{if}$$

$$1-\beta-(4+2\alpha\beta-6\alpha)\, r + (2\alpha-1)\,(2\alpha-\beta-1)r^2 \geq 0.$$

and

(2.4.8) $$a'^2 + \{2\beta(1-\alpha) - 2\alpha\,(3-2\alpha)\}\, a' + (\alpha+\beta(1-\alpha))^2 \geq 0$$

(2.4.7) is valid only when $0 \leq r \leq r_0 \leq r'(\alpha)$ and

(2.4.8) is valid only when $r'(\alpha) \leq r \leq r'_0 < 1.$

The equality in (2.4.7) is attained for the function given by

(2.4.9) $$f(z) = z(1-z)^{2\alpha-2}$$

and that in (2.4.8) for the function given by

(2.4.10) $$f(z) = z(1-2bz + z^2)^{-1+\alpha}$$

where b is determined by

$$\frac{1 - 2\alpha br + (2\alpha-1)\, r^2}{1 - 2br + r^2} = \sqrt{\frac{\alpha a'}{2-\alpha}} = R'_0 \qquad \text{(say)}$$

It is easy to check that function given by (2.4.10) is regular in D because $b \in (-1,1)$ can be easily verified.

By taking $\beta = 0$ in (2.4.2) and (2.4.3) we obtain Theorem 2 of V.A. Zmorsovič [39] as special case of Theorem 2.

2.5 Inequalities for class R.

Notations:

Let $$\mu(r,\alpha) = \frac{1 + (2\alpha-1)r}{1 + r}$$

$$\mathrm{Re}\, \left\{\frac{zh'(z)}{h(z)}\right\} \geq \sigma(r,\alpha)$$

where

$$\sigma(r,\alpha) = \begin{cases} \dfrac{-2r(1-\alpha)}{(1+r)\ (1+(2\alpha-1)r)} = \sigma_1(r,\alpha) \\ \text{for } 0 \le r \le r(\alpha), \\ -\dfrac{(\sqrt{1-(2\alpha-1)r^2} - \sqrt{\alpha(1-r^2)})^2}{(1-\alpha)\ (1-r^2)} = \sigma_2\ (r,\alpha) \\ \text{for } r(\alpha) \le r < 1. \end{cases}$$

and $\eta(r,\alpha) = \dfrac{2r(1-\alpha)}{(1-r)\ (1-(2\alpha-1)r)}$

Theorem 3: Let $f'(z) = g'(z)\ h(z)$, where $g'(z) \in P(\alpha)$, $0 \le \alpha < 1$, and $h(z) \in P(\beta)$, $0 \le \beta < 1$, and $\beta \le \alpha$, then $f(z)$ is convex for $|z| < r_o < 1$, where r_o the smallest positive root of the equation.

$$(2.5.1)\quad T(r,\alpha,\beta) \equiv \left[\begin{array}{l} 1+\sigma_1(r,\alpha)+\sigma_1(r,\beta),\ 0 \le r \le r(\alpha) \\ 1+\sigma_2(r,\alpha)+\sigma_1(r,\beta),\ r(\alpha) \le r \le r(\beta) \\ 1+\sigma_2(r,\alpha)+\sigma_2(r,\beta),\ r(\beta) \le r < 1 \end{array}\right] = 0$$

where $r(\alpha)$ is the smallest positive root of the equation

$$(1-2\alpha)\ r^3 - 3(1-2\alpha)\ r^2 + 3r - 1 = 0.$$

This result is sharp.

Proof : $f'(z) = h(z)\ g'(z)$

Logarithmic diff. of $f'(z)$ yields

$$(2.5.2)\qquad \text{Re}\ \{1 + \frac{zf''(z)}{f'(z)}\} = 1 + \text{Re}\ \{\frac{zg''(z)}{g'(z)}\} + \text{Re}\ \{\frac{zh'(z)}{h(z)}\}$$

Applying lemma 3 to (2.5.2), we have

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \begin{cases} 1+\sigma_1(r,\alpha)+\sigma_1(r,\beta), & 0 \leq r \leq r(\alpha) \\ 1+\sigma_2(r,\alpha)+\sigma_1(r,\beta), & r(\alpha) \leq r \leq r(\beta) \\ 1+\sigma_2(r,\alpha)+\sigma_2(r,\beta), & r(\beta) \leq r < 1. \end{cases}$$

As $g_1(r,\alpha)$ is a strictly increasing function of α, $0 \leq \alpha < 1$, for any r, $0 \leq r \leq 1$.

$$g(r,\beta) \leq g(r,\alpha)$$

Hence

$$g(r(\alpha),\beta) \leq g\,(r\,(\alpha),\alpha) = 0$$

$$g(r(\alpha),\beta) \leq 0.$$

Therefore

$$r(\alpha) \leq r(\beta)$$

Since $T(0,\alpha,\beta) = 1$

and $\lim\limits_{r \to 1} T(r,\alpha,\beta) = -\infty$, thus the equation (2.5.1) has a root always in $(0,1)$. The sharpness of the theorem follows from the sharpness of the results of the lemma 3. Proofs of the following theorems are similar to the theorem 3 hence omitted.

Theorem 4 : *Let* $f'(z) = g'(z)\,h(z)$, *where* $g(z) \in C(\alpha)$, $0 \leq \alpha < 1$ *and* $h(z) \in P(\beta)$, $0 \leq \beta < 1$, *then* $f(z)$ *is convex for* $|z| < r_0 < 1$, *where* r_0 *is the smallest positive root of the equation*

$$T(r,\alpha,\beta) = \begin{bmatrix} \mu(r,\alpha) + \sigma_1(r,\beta), 0 \leq r \leq r(\beta) \\ \\ \mu(r,\alpha) + \sigma_2(r,\beta),\ r(\beta) \leq r < 1 \end{bmatrix} = 0$$

where $r(\beta)$ is the smallest positive root of the equation

$$(1-2\beta)\, r^3 - 3(1-2\beta)\, r^2 + 3r - 1 = 0.$$

This result is sharp.

Theorem 5 : Let $f'(z) = \frac{g'(z)}{h(z)}$, where $g(z) \in C(\alpha)$, $0 \le \alpha < 1$ and $h(z) \in P(\beta)$, $0 \le \beta < 1$, then $f(z)$ is convex for $|z| < r_o < 1$, where r_o is the smallest positive root of the equation.

$$\mu(r,\alpha) - \eta(r,\beta) = 0.$$

This result is sharp.

Remark : Theorems 2,3 of Ratti [30] follow from our Theorem 3 by taking $\beta = 0$, $\alpha = 0$; $\alpha = 1/2$, $\beta = 0$ respectively. Also his Theorem 6 follows from the Theorem 4 by taking $\alpha = 1/2$, $\beta = 0$.

2.6 Radius of starlikeness for the class M.

Notations : Let

$$\theta_1(r,\alpha) = \frac{1-(2-4\alpha)\, r \pm \alpha(2\alpha-1)\, r^2}{(1+r)\,(1+\alpha r)}$$

$$\theta_2(r,\alpha) = \frac{2}{1-\alpha}\{\sqrt{(1+\alpha)\,(4-2\alpha)a} - 1 - a\}.$$

Theorem 6: Let $f(z) = \frac{zF(z)}{2g(z)}\left[1+\frac{zF'(z)}{F(z)}\right]$, where $F(z) \in S^*(\alpha)$, $0 \le \alpha < 1$, $g(z) \in C$, then $f(z)$ is starlike for $|z| < r_o < 1$, where r_o is the smallest positive root of the equation.

$$T(r,\alpha) = \begin{bmatrix} 1 - \frac{1}{1-r} + \theta_1(r,\alpha), & 0 \le r \le r_1(\alpha) \\ 1 - \frac{1}{1-r} + \theta_2\,(r,\alpha), & r_1(\alpha) \le r < 1 \end{bmatrix} = 0$$

where $r(\alpha)$ is the smallest positive root of the equation

$$\alpha(2\alpha-1)\ r^3 + \alpha(7-2\alpha)r^2 + (5-4\alpha)r - 3 = 0$$

This result is sharp.

Proof : Since $f(z) \in S^*(\alpha)$, $\frac{zF'(z)}{F(z)} \in P(\alpha)$, therefore

$$f(z) = \frac{zF(z)}{g(z)}\ \left[\frac{1+\alpha z\phi(z)}{1+z\phi(z)}\right] .$$

Logarithmic diff. of $f(z)$ yields

$$\frac{zf'(z)}{f(z)} = 1 - \frac{zg'(z)}{g(z)} + \frac{1+(2\alpha-1)z\phi(z)}{1+z\phi(z)} - \frac{(1-\alpha)\ (z^2\phi'(z) + z\phi(z))}{(1+z\phi(z))\ (1+\alpha z\phi(z))}$$

$$(2.6.1)\quad \mathrm{Re}\{\frac{zf'(z)}{f(z)}\} \geq \begin{cases} 1 - \frac{1}{1-r} + \theta_1(r,\alpha), & 0 \leq r \leq r_1(\alpha) \\ 1 - \frac{1}{1-r} + \theta_2\ (r,\alpha), & r_1(\alpha) \leq r < 1. \end{cases}$$

(2.6.1) has been obtained by using lemma 5 and the fact that

$$\left|\frac{zg'(z)}{g(z)}\right| \leq \frac{1}{1-r}$$

Since bounds used here are sharp, hence the results of this theorem are sharp:

Proofs of the following theorems are similar to theorem 6, hence omitted

Theorem 7: Let $f(z) = \frac{zF(z)}{g(z)}\ \ [1 + \frac{zF'(z)}{F(z)}]$, where $F(z) \in S^*(\alpha)$ $0 \leq \alpha < 1$, $g(z) \in S^*(\beta)$, $0 \leq \beta < 1$, then $f(z)$ is starlike for $|z| < r_0 < 1$, where r_0 is the smallest positive root of the equation

$$T(r,\alpha,\beta) = \begin{bmatrix} \mu(r,\beta) + \theta_1(r,\alpha) + 1, & 0 \leq r \leq r_1(\alpha) \\ \mu(r,\beta) + \theta_2(r,\alpha) + 1, & r_1(\alpha) \leq r < 1 \end{bmatrix}$$

$$= 0 ,$$

where $r(\alpha)$ is same as stated in Theorem 6. This result is sharp.

Theorem 8: Let $f(z) = \frac{zF(z)}{g(z)}\ [1 + \frac{zF'(z)}{F(z)}]$, where $F(z) \in S^*(\alpha)$, $0 \leq \alpha < 1$ and $\frac{g(z)}{z} \in P(\beta)$, $0 \leq \beta < 1$, then $f(z)$ is starlike for $|z| < r_o < 1$, where r_o is the smallest positive root of the equation

$$T(r,\alpha,\beta) = \left[\begin{array}{l} \theta_1(r,\alpha) - \eta(r,\beta), 0 \leq r \leq r(\alpha) \\ \\ \theta_2(r,\alpha) - \eta(r,\beta), r(\alpha) \leq r < 1 \end{array}\right]$$

$$= 0$$

where $r(\alpha)$ is the same as stated in the theorem 6. This result is sharp.

Remark : By taking $\alpha = \beta = 0$ in Theorems 6,7, and 8, Theorems 1,2, and 3 of E.G. Calys [6] follow immediately.

2.7 Radius of convexity for the class $H(\alpha)$

Theorem 9: Let $f(z) \in H(\alpha)$, then $f(z)$ is convex for $|z| < r_o$, where

$$r_o = \left[\begin{array}{ll} \frac{3-\sqrt{5}}{2\alpha} & \text{if } \alpha_o \leq \alpha < 1, \\ \\ \left[\frac{2(1+\alpha)^2 - 3\alpha - 2(1-\alpha)\sqrt{\alpha^2+4\alpha+1}}{\alpha(4\alpha-5)}\right]^{1/2} & \text{if } 0 < \alpha \leq \alpha_o \end{array}\right.$$

$$\alpha_o = \frac{3-\sqrt{5} + 2\sqrt{3(7-3\sqrt{5})}}{2\sqrt{5}} \approx .589.$$

This result is sharp.

Proof : Since $f(z) \in H(\alpha)$, there exists a function $w(z) = z\phi(z)$ satisfying Schwarz's lemma such that

$$(2.7.1) \qquad \frac{zf'(z)}{f(z)} = 1 + \alpha z\phi(z).$$

Diff. (2.7.1) logarithmically with respect to z and then combining it with (2.7.1) we get

$$(2.7.2) \qquad 1+\frac{zf''(z)}{f'(z)} = 1+\alpha z\phi(z) + \frac{\alpha(z^2\phi'(z)+z\phi(z))}{(1+\alpha z\phi(z)}$$

Therefore

$$(2.7.3) \qquad \mathrm{Re}\ \{1+\frac{zf''(z)}{f'(z)}\} \geq 2+\alpha\ \mathrm{Re}\ (z\phi(z)) - \mathrm{Re}\{\frac{1}{1+\alpha z\phi(z)}\} - \frac{\alpha(|z|^2-|z\phi(z)|^2)}{(1-|z|^2)|(1+\alpha z\phi(z))}$$

Let $|z| = r$ and $1+\alpha z\phi(z) = u+iv$, then

$$(2.7.4) \qquad \mathrm{Re}\ \{1+\frac{zf''(z)}{f'(z)}\} \geq 1+u-\frac{u}{u^2+v^2} - \frac{\alpha^2r^2-(u-1)^2-v^2}{\alpha(1-r^2)\ \sqrt{u^2+v^2}} \equiv H(u,v,r)\ (\text{say}).$$

Diff. of H partially w.r.t. v yields

$$(2.7.5) \quad \frac{\partial H}{\partial v} = v\left[\frac{2u}{(u^2+v^2)^2} + \frac{2}{\alpha(1-r^2)\ \sqrt{(u^2+v^2)}} + \frac{\alpha^2r^2-(u-1)^2-v^2}{\alpha(1-r^2)\ (u^2+v^2)^{3/2}}\right]$$

Since $|z\phi(z)| \leq r$ and $|z\phi(z)|^2 = \frac{(u-1)^2+v^2}{\alpha^2}$,

we can easily conclude that

$$(2.7.6) \qquad \alpha^2r^2-(u-1)^2-v^2 \geq 0$$

and from $1+\alpha z\phi(z) = u+iv$

$$(2.7.7) \qquad 1-\alpha r \leq u \leq 1+\alpha r$$

(2.7.6) ensures us that the quantity within square brackets of (2.7.5) is strictly positive, therefore minimum of H occurs at $v = 0$. On putting $v = 0$ in (2.7.4), we obtain

$$h(u,r) \equiv H(u,0,r) = 1+u-\frac{1}{u} - \frac{\alpha^2r^2-(u-1)^2}{\alpha(1-r^2)u}$$

$$= 1 - \frac{2}{\alpha(1-r^2)} + \frac{u(\alpha+1-\alpha r^2)}{\alpha(1-r^2)} + \frac{(1-\alpha)\ (1+\alpha r^2)}{\alpha(1-r^2)u}$$

Absolute minimum of h in $(0,\infty)$ is attained at

$$(2.7.8)\quad u_o = \sqrt{\frac{(1-\alpha)\ (1+\alpha r^2)}{\alpha+1-\alpha r^2}}$$

and is equal to

$$(2.7.9)\quad h(u_o,r) \equiv \frac{\alpha(1-\alpha r^2) - 2 + 2\sqrt{(1-\alpha)(\alpha+1-\alpha r^2)(1+\alpha r^2)}}{\alpha(1-r^2)}$$

provided $u_o \in [1-\alpha r,\ 1+\alpha r]$. One can easily check that $u_o < 1+\alpha r$ but it is not always greater than $1-\alpha r$. In such a case when $u_o \notin [1-\alpha r,\ 1+\alpha r]$ minimum of h on the segment $[1-\alpha r,\ 1+\alpha r]$ is attained at

$$(2.7.10)\quad u_1 = 1-\alpha r$$

and is equal to

$$(2.7.11)\quad h(u_1,r) = \frac{1-3\alpha r+\alpha^2 r^2}{1-\alpha r}.$$

$h(u_1,r) = h(u_o,r)$ for such values of α for which

$$(2.7.12)\quad u_o = u_1$$

Therefore

$$\mathrm{Re}\ \{1+\frac{zf''(z)}{f'(z)}\} \geq \frac{1-3\alpha r+\alpha^2 r^2}{1-\alpha r}\quad \text{if } u_o \leq u_1,$$

$$\mathrm{Re}\ \{1+\frac{zf''(z)}{f'(z)}\} \geq \frac{\alpha(1-r^2) - 2 + 2\sqrt{(1-\alpha)\ (\alpha+1-\alpha r^2)(1+\alpha r^2)}}{\alpha(1-r^2)}\quad \text{if } u_o \geq u_1$$

Therefore $\mathrm{Re}\ \{1+\frac{zf''(z)}{f'(z)}\} \geq 0$ if

$$(2.7.13)\quad 1-3\alpha r+\alpha^2 r^2 \geq 0$$

and

(2.7.14) $\quad r^4\alpha\ (4\alpha-5) + r^2(6-4\alpha-4\alpha^2) - 5\alpha+4 \geq 0$

The smallest positive roots of (2.7.13) and (2.7.14) lie in (0,1) and become equal to each other for such values of α for which

(2.7.15) $\quad u_0 = u_1$

Eliminating r between (2.7.13) and (2.7.15) we obtain

$$\alpha_0 = \frac{3-\sqrt{5} + 2\sqrt{3(7-3\sqrt{5})}}{2\sqrt{5}} \approx .589$$

Therefore $f(z)$ is convex for

(2.7.16) $\quad |z| < \dfrac{3-\sqrt{5}}{2\alpha} \quad \text{if } \alpha_0 \leq \alpha \leq 1$

and for

(2.7.17) $\quad |z| < \left[\dfrac{2(1+\alpha^2) - 3\alpha - 2(1-\alpha)\sqrt{\alpha^2+4\alpha+1}}{\alpha(4\alpha-5)}\right]^{1/2}$

if $0 < \alpha \leq \alpha_0$.

Equality sign in (2.7.13) is attained for the function

$$f(z) = ze^{\alpha z} \quad , \alpha_0 \leq \alpha \leq 1$$

and that in (2.7.14) for function

$$f(z) = z\,e^{\left[\alpha\int_0^z (\beta-t)(1-\beta t)dt\right]} \qquad \text{if } 0 < \alpha \leq \alpha_0$$

By a Theorem of D.J. Wright [37] $f(z) \in H(\alpha)$ provided

(2.7.8) $-1 \leq \beta \leq 1$.

Let

$$\beta = \frac{2 - 3\alpha + \alpha r^2\ (3 - 2\alpha)}{2r\ (\alpha - 1)^2}$$

It is easily verified that for $0 < \alpha \leq \alpha_0$ and for r given by (2.7.17), β fulfils the required condition (2.7.18).

<u>Remark</u> : Recently the above result has been obtained by P.J. Eenigenburg [8] by using different techniques.

CHAPTER 3

RADII OF CONVEXITY AND STARLIKENESS OF SOME CLASSES OF MEROMORPHIC FUNCTIONS

3.1 Let $\Sigma^*(\alpha)$ denote the class of functions

(3.1.1) $$f(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + \ldots\ldots$$

which are regular in D except a pole at $z = 0$ and satisfy the condition

(3.1.2) $$-\operatorname{Re} \left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad , \; z \in D.$$

Let $M(\alpha)$ stands for the class of functions of the form (3.1.1) which satisfy the condition

(3.1.3) $$-\operatorname{Re} \left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad , \; z \in D$$

Functions in the classes $\Sigma^*(\alpha)$ and $M(\alpha)$ are known as meromorphic starlike and convex functions of order α respectively. $P(\alpha)$ denote the class of functions which are regular in D and have real part greater than α there.

Let H_α be the class of functions

(3.1.4) $$f(z) = -\frac{z^2}{2} \left[\frac{F(z)}{z}\right]',$$

where $F(z) \in \Sigma^*(\alpha)$

Let T denote the class of functions

$$(3.1.5) \qquad f(z) = s(z)\, p(z)\, q(z)$$

where $s(z) \in \Sigma^*(\alpha)$, $p(z) \in P(\beta)$ and $q(z) \in P(\gamma)$.

This chapter deals with the determination of radius of convexity for the class $\Sigma^*(\alpha)$ and radii of starlikeness for classes H_α and T.

3.2 <u>We start by proving following lemmas.</u>

<u>Lemma</u> 1 : Let

$$(3.2.1) \quad -H(z) = \frac{a}{1+z\phi(z)} + \frac{1}{1+bz\phi(z)} + \frac{(1-b)z^2\phi'(z)}{(1+z\phi(z))\ (1+bz\phi(z))}, -1 \leq a,b \leq 1$$

where $\phi(z)$ is regular, $|\phi(z)| \leq 1$ in D.

Then for $|z| = r$, $0 \leq r < 1$,

$$(3.2.2) \quad -\mathrm{Re}\{H(z)\} \leq \frac{3-ab - 2br^2 - r^2 + abr^2}{(1-b)\ (1-r^2)} - \frac{2}{1-b}\sqrt{(1+b)(1-a)\ a'}$$

where $$a' = \frac{1-br^2}{1-r^2}$$

$$(3.2.3) \quad -\mathrm{Re}\{H(z)\} \geq \begin{cases} -\dfrac{1+ab+(1-2b-ab)r^2}{(1-b)(1-r^2)} + \dfrac{2}{(1-b)}\sqrt{\dfrac{(1+a)(1-b)(1+br^2)}{1-r^2}} \\ \text{if } u_o \leq u_1, \\ \dfrac{1 + a - (1+ab)r}{(1-r)(1-br)} \ \text{if } u_o \geq u_1 \end{cases}$$

<u>where</u> $u_o = \frac{1}{1-b}\{\sqrt{\frac{(1-b)(1+br^2)}{(1+a)\ (1-r^2)}} - b\}$, $u_1 = \frac{1}{1-r}$.

<u>Proof</u> : From (3.2.1) we have

$$(3.2.4) \quad -\mathrm{Re}\ \{H(z)\} \leq a\ \mathrm{Re}\{\frac{1}{1+z\phi(z)}\} + \mathrm{Re}\{\frac{1}{1+bz\phi(z)}\}$$

$$+ \frac{(1-b)\{|z|^2 - |z\phi(z)|^2\}}{(1-|z|^2)\,|1+bz\phi(z)|\ \ |1+z\phi(z)|}.$$

The above inequality has been obtained by using (2.2.5).

Let $|z| = r$ and $\frac{1}{1+z\ \phi(z)} = u + iv$, then u satisfies

$$\left|u - \frac{1}{1-|z|^2}\right| \leq \frac{|z|}{1-|z|^2}$$

i.e.,

$$(3.2.5) \qquad \frac{1}{1+r} \leq u \leq \frac{1}{1-r}$$

Putting values of $|z|$ and $\frac{1}{1+z\phi(z)}$ in (3.2.4), we obtain

$$(3.2.6) \quad -\mathrm{Re}\ \{H(z)\} \leq [au + \frac{bu + (1-b)\ u^2 + (1-b)\ v^2}{(b+(1-b)u)^2 + (1-b)^2\ v^2}$$

$$+ \frac{(1-b)\ \{r^2(u^2+v^2) - (1-u)^2 - v^2\}}{(1-r^2)\sqrt{(b+(1-b)u)^2 + (1-b)^2\ v^2}}]$$

$$\equiv G(u,v,r) \quad (\text{say.})$$

Diff. G partially w.r.t. v we have

$$(3.2.7)\quad \frac{\partial G}{\partial v} = -v\Big[-\frac{2b(1-b)(b+(1-b)u)}{\{(b+(1-b)u)^2+(1-b)^2v^2\}^2}+\frac{2(1-b)}{\sqrt{(b+(1-b)u)^2+(1-b)^2v^2}}$$
$$+\frac{\{r^2(u^2+v^2)-(1-u)^2-v^2\}(1-b)^3}{(1-r^2)\{(b+(1-b)u)^2+(1-b)^2v^2\}^{3/2}}\Big]$$

Since $|z\phi(z)| \leq r$ for $|z| = r$ and $|z\phi(z)|^2 = \dfrac{(1-u)^2+v^2}{u^2+v^2}$, we can easily conclude that

$$(3.2.8)\qquad r^2(u^2+v^2)-(1-u)^2-v^2 \geq 0.$$

(3.2.5) and (3.2.8) guarantee that quantity within square brackets of (3.2.7) is strictly positive, therefore maximum of G occurs at $v = 0$. On putting $v = 0$ in (3.2.6) we obtain

$$g(u,r) \equiv G(u,0,r) = \frac{2}{1-r^2}+\frac{1+b}{1-b}+(a-1)u+\frac{(1+b)(1-br^2)}{(1-r^2)(1-b)(b+(1-b)u)}$$

Absolute maximum of g in $(0,\infty)$ is attained at

$$b+(1-b)u_0' = \sqrt{\frac{(1+b)(1-br^2)}{(1-a)(1-r^2)}} = \sqrt{\frac{(1+b)a'}{1-a}}$$

i.e.,

$$(3.2.9)\qquad u_0' = \frac{1}{1-b}\Big\{\sqrt{\frac{(1+b)a'}{1-a}}-b\Big\}$$

where $a' = \dfrac{1-br^2}{1-r^2}$

and is equal to

Diff. G partially w.r.t. v we have

$$(3.2.7)\quad \frac{\partial G}{\partial v} = - v\ [- \frac{2b(1-b)(b+(1-b)u)}{\{(b+(1-b)u)^2+(1-b)^2v^2\}^2} + \frac{2(1-b)}{\sqrt{(b+(1-b)u)^2+(1-b)^2v^2}} + \frac{\{r^2(u^2+v^2) - (1-u)^2 - v^2\}\ (1-b)^3}{(1-r^2)\ \{(b+(1-b)u)^2 + (1-b)^2\ v^2\}^{3/2}}]$$

Since $|z\phi(z)| \leq r$ for $|z| = r$ and $|z\phi(z)|^2 = \frac{(1-u)^2+ v^2}{u^2 + v^2}$, we can easily conclude that

$$(3.2.8)\qquad r^2\ (u^2+ v^2) - (1-u)^2- v^2 \geq 0.$$

(3.2.5) and (3.2.8) guarantee that quantity within square brackets of (3.2.7) is strictly positive, therefore maximum of G occurs at $v = 0$. On putting $v = 0$ in (3.2.6) we obtain

$$g(u,r) \equiv G(u,0,r) = \frac{2}{1-r^2} + \frac{1+b}{1-b} + (a-1)\ u + \frac{(1+b)\ (1-br^2)}{(1-r^2)(1-b)(b+(1-b)u)}$$

Absolute maximum of g in $(0, \infty)$ is attained at

$$b + (1-b)\ u_o' = \sqrt{\frac{(1+b)\ (1-br^2)}{(1-a)\ (1-r^2)}} = \sqrt{\frac{(1+b)\ a'}{1-a}}$$

i.e.,

$$(3.2.9)\qquad u_o' = \frac{1}{1-b}\ \{\sqrt{\frac{(1+b)a'}{1-a}} - b\ \}$$

where $a' = \frac{1-br^2}{1-r^2}$

and is equal to

$$g(u_o',r) \equiv \frac{3 - ab - 2br^2 - r^2 + abr^2}{(1-b)\ (1-r^2)} - \frac{2}{1-b}\ \sqrt{(1+b)(1-a)a'}$$

Thus the proof of the first part is complete.

Now using (2.2.5) in (3.2.1) together with $|z| = r$ and $\frac{1}{1+z\phi(z)} = \frac{ue^{iv}-b}{1-b}$, we have

$$(3.2.10)\quad -\operatorname{Re}\{H(z)\} \geq \frac{\cos v}{1-b}\left(au - \frac{b}{u} - \frac{2(1-br^2)}{(1-r^2)}\right) + \frac{1-ab}{1-b} + \frac{1}{1-b}\left(\frac{1-b\ r^2}{(1-r^2)u} + u\right) \equiv L(u,v,r)\ .$$

Diff. L partially w.r.t. v we have

$$(3.2.11)\qquad \frac{\partial L}{\partial v} = \frac{\sin v}{1-b}\left(-au + \frac{b}{u} + \frac{2(1-br^2)}{1-r^2}\right) = M(u)\ \frac{\sin v}{(a-b)}$$

$$M(u) \geq -\left(u + \frac{1}{u}\right) + \frac{2(1-br^2)}{1-r^2}$$

The minimum of $-(u + \frac{1}{u}) + \frac{2(1-br^2)}{1-r^2}$ in $[\frac{1+br}{1+r}, \frac{1-br}{1-r}]$ is attained at end points only. The value of $-(u + \frac{1}{u}) + \frac{2(1-br^2)}{1-r^2}$ is positive at end points. Hence minimum of L occurs at $v = 0$. Putting $v = 0$ in (3.2.10) we obtain

$$\ell(u,r) \equiv L(u,0,r) = \frac{1}{1-b}\left(au\ \frac{-b}{u} - \frac{2(1-br^2)}{1-r^2}\right) + \frac{1-ab}{1-b} + \frac{1}{(1-b)}\left(\frac{1-br^2}{(1-r^2)u} + u\right)$$

$$= \frac{1}{1-b}\left[(a+1)\ u\ \frac{1-b}{u(1-r^2)}\right] - \frac{2(1-br^2)}{(1-b)(1-r^2)} + \frac{1-ab}{1-b}$$

Absolute minimum of ℓ in $(0, \infty)$ is attained at

$$u_o = \sqrt{\frac{(1-b)}{(1+a)(1-r^2)}}$$

and is equal to

$$\ell(u_o,r) = \frac{1}{(1-b)} [2\sqrt{(1-b)(1+a)(1-r^2)} + \frac{(1-ab)(1-r^2) - 2(1-br^2)}{(1-b)(1-r^2)}]$$

It is easy to check that $u_o > \frac{1+br}{1+r}$ but u_o is not always less than $\frac{1-br}{1-r}$. In such a case, when $u_o \notin [\frac{1+br}{1+r}, \frac{1-br}{1-r}]$, minimum of ℓ on the segment $[\frac{1+br}{1+r}, \frac{1-br}{1-r}]$ is attained at

$$u = u_1 = \frac{1-br}{1-r}$$

and is equal to $\frac{1+a-(1+ab)r}{(1-br)(1-r)}$.

$\ell(u_o,r) = \ell(u_1,r)$ for such values of a and b for which

$$u_o = u_1$$

<u>Lemma</u> 2 : If $f(z) \in \Sigma^*(\alpha)$, then for $|z| = r$, $0 \leq r < 1$,

$$(3.2.12) \qquad -\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \leq \frac{a-2\alpha\sqrt{a}+\alpha}{1-\alpha}$$

$$(3.2.13) \quad -\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \begin{cases} 2\sqrt{\dfrac{1-2\alpha+\alpha a}{1-\alpha}}+\dfrac{\alpha-a}{1-\alpha} & \underline{\text{for}}\ u_o \leq u_1, \\[2ex] \dfrac{1-2\alpha r+(2\alpha-1)^2 r^2}{(1-r)\ (1-(2\alpha-1)r)} & \underline{\text{for}}\ u_o \geq u_1 \end{cases}$$

<u>where</u> $u_o = \frac{1}{2(1-\alpha)}\left\{\sqrt{\frac{1-2\alpha+\alpha a}{1-\alpha}}-(2\alpha-1)\right\},\ u_1 = \frac{1}{1-r}$

<u>and</u> $a = \frac{1-(2\alpha-1)r^2}{1-r^2}$. <u>The results are sharp</u>.

<u>Proof</u> : Since $f(z)\varepsilon\Sigma^*(\alpha)$ there exists a function $w(z) = z\phi(z)$ satisfying Schwarz's lemma such that

$$(3.2.14) \qquad -\frac{zf'(z)}{f(z)} = \frac{1+(2\alpha-1)\ z\phi(z)}{1-z\phi(z)}$$

Diff. (3.2.14) logarithmically w.r.t to z we have

$$-\left(1+\frac{zf''(z)}{f'(z)}\right) = 2\alpha-1+\frac{1-2\alpha}{1+z\phi(z)}+\frac{1}{1+(2\alpha-1)z\phi(z)} + \frac{2(1-\alpha)\ z^2\phi'(z)}{(1+z\phi(z))\ (1+(2\alpha-1)\ z\phi(z)}$$

taking $a = 1-2\alpha$, $b = (2\alpha-1)$ in (3.2.2) and in (3.2.3) we obtain (3.2.12) and (3.2.13) respectively.

The equality sign in (3.2.12) is attained for the function $f(z)$ given by

$$-\frac{zf'(z)}{f(z)} = \frac{1-2\alpha\, bz + (2\alpha-1)\, z^2}{1 - 2bz + z^2},$$

where b is determined from

$$\frac{1-2\alpha\, br + (2\alpha-1)r^2}{1-2br + r^2} = \left[\frac{1-(2\alpha-1)r^2}{1-r^2}\right]^{1/2}$$

The equality signs in (3.2.13) for the second and first inequalities are attained respectively for the functions defined by the following equations

$$(3.2.15) \qquad f(z) = \frac{(1-z)^{2-2\alpha}}{z}.$$

and

$$(3.2.16) \qquad f(z) = \frac{1}{z}\, \{(1-z)^{1+m}\, (1+z)^{1-m}\}^{1-\alpha}$$

where m is determined from

$$\left[\frac{1+(2\alpha-1)r^2}{1-r^2}\right]^{1/2} = \frac{1-(2\alpha-1)\, r^2 + 2m(1-\alpha)r}{1-r^2}$$

Remark : The above result has also been obtained by V. Singh and R.M. Goel [34] .

Lemma 3 : If $f(z) \in H_\alpha$, then for $|z| = r$, $0 \leq r < 1$,

$$(3.2.17) \qquad -\text{Re}\, \left\{\frac{zf'(z)}{f(z)}\right\} \leq \frac{2}{1-\alpha}\, \{a + \alpha - \sqrt{2\alpha(1+\alpha)\, a}\}$$

$$(3.2.18)\quad -\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \begin{cases} \dfrac{2}{1-r^2}\left\{\sqrt{\dfrac{2(1+\alpha r^2)}{(1-r^2)}}-1\right\} \\ \text{if } u_o \leq u_1 \\ \dfrac{1-2\alpha r+\alpha(2\alpha-1)r^2}{(1-r)(1-\alpha r)} \quad \text{if } u_o \geq u_1 \end{cases}$$

<u>where</u> $a = \dfrac{1-\alpha r^2}{1-r^2}$, $u_o = \dfrac{1}{1-\alpha}\left\{\sqrt{\dfrac{1+\alpha r^2}{2(1-r^2)}} - b\right\}$ <u>and</u> $u_1 = \dfrac{1}{1-r}$.

<u>Proof</u> : Since $f(z) \in H_\alpha$ we have

$$f(z) = F(z)\,\frac{(1+\alpha z\phi(z))}{1+z\,\phi(z)}, \quad z \in D$$

Diff. $f(z)$ logarithmically w.r.t z, w have

$$-\frac{zf'(z)}{f(z)} = 2\alpha-1 + \frac{1-2\alpha}{1+z\phi(z)} + \frac{1}{1+\alpha z\,\phi(z)} + \frac{(1-\alpha)\, z^2\phi'(z)}{(1+z\phi(z))(1+\alpha z\phi(z))}$$

Taking $a = 1-2\alpha$, $b = \alpha$ in (3.2.2) and (3.2.3) we get (3.2.17) and (3.2.18) respectively. The equality sign in (3.2.17) is attained for the function $F(z)$ given by

$$-\frac{zF'(z)}{F(z)} = \frac{1-2\alpha bz + (2\alpha-1)z^2}{1-2bz+z^2}$$

where b is determined from

$$\frac{1-(1+\alpha)\,br+\alpha r^2}{1-2br+r^2} = \sqrt{\frac{1-\alpha r^2}{1-r^2}}\,.$$

The signs of equality in (3.2.18) for the second and first inequalities are attained respectively for functions $F(z)$ given by (3.2.15) and

$$(3.2.19) \qquad F(z) = \frac{1}{z}\{(1-z)^{1+m}(1+z)^{1-m}\}^{1-\alpha},$$

where m is determined from

$$\sqrt{\frac{1+\alpha r^2}{2(1-r^2)}} = \frac{1-\alpha r^2+m(1-\alpha)r}{1-r^2}$$

3.3 Radius of convexity for the class $\Sigma^*(\alpha)$.

Theorem 1: Let $f(z) \varepsilon \Sigma^*(\alpha)$. Let x_o be the unique positive root of the equation

$$x^4 - 4x^3 + 2x^2 - 8 = 0$$

and $\alpha_o = \dfrac{x_o^2-4}{4x_o-4}$, then $f(z)$ is convex of order β, $0 \leq \beta < 1$, in

$$0 < |z| < \left[\frac{(4\alpha-5+2\beta^2)+4\sqrt{(1-\alpha+\alpha^2-\alpha\beta)+\frac{\beta^2}{16}(3\beta^2+8\alpha-10)}}{8\alpha-4+(1-\beta)^2}\right]^{1/2}$$

if $0 \leq \alpha \leq \alpha_o$,

$$0 < |z| < \frac{1-\beta}{\alpha(1-\beta)+\sqrt{(1-\beta)(1-\alpha)(3\alpha-1-\beta(1-\alpha))}}$$

if $\alpha_o \leq \alpha < 1$.

Proof : Since $f(z) \varepsilon \Sigma^*(\alpha)$ from lemma 2 we have

$$(3.3.1) \qquad -\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}+\beta\right\} \geq \frac{1-2\alpha r+(2\alpha-1)^2r^2}{(1-r)(1-(2\alpha-1)r)} - \beta$$

$$= \frac{1-\beta-2\alpha(1-\beta)r+(2\alpha-1)(2\alpha-\beta-1)r^2}{(1-r)(1-(2\alpha-1)r)}$$

if $u_0 \geq u_1$,

and

$$(3.3.2) \qquad -\mathrm{Re}\,\{1+\frac{zf''(z)}{f'(z)}+\beta\} \geq 2\sqrt{\frac{1+(2\alpha-1)r^2}{1-r^2}} - \frac{1+r^2}{1-r^2} - \beta$$

$$= 2\sqrt{\frac{1+(2\alpha-1)r^2}{1-r^2}} - \frac{1+\beta+(1-\beta)r^2}{1-r^2}$$

if $u_0 \leq u_1$

where $u_0 = \dfrac{1}{2(1-\alpha)} \left\{\sqrt{\dfrac{1+(2\alpha-1)r^2}{1-r^2}} - 2\alpha + 1\right\}$

and $u_1 = \dfrac{1}{1-r}$

Therefore

$$-\mathrm{Re}\,\{1+\frac{zf''(z)}{f'(z)}+\beta\} \geq 0$$

if

$$(3.3.3) \qquad 1-\beta - 2\alpha(1-\beta)\,r + (2\alpha-1)\,(2\alpha-\beta-1)\,r^2 \geq 0$$

$$(3.3.4) \qquad \{8\alpha - 4 + (1-\beta)^2\}\,r^4 - 2(4\alpha-5+2\beta^2)r^2 - 4 + (1+\beta)^2 \geq 0.$$

Therefore radii of convexity for the class $\Sigma^*(\alpha)$ are given by

$$(3.3.5) \qquad r = \frac{1-\beta}{\alpha(1-\beta) + \sqrt{(1-\beta)(1-\alpha)\,\{3\alpha-1-\beta(1-\alpha)\}}}$$

$$(3.3.6) \qquad r = \left[\frac{(4\alpha-5+2\beta^2) + 4\sqrt{(1-\alpha+\alpha^2-\alpha\beta)+\frac{1}{16}\,(3\beta^2+8\alpha-10)}}{8\alpha-4+(1-\beta)^2}\right]^{1/2}$$

The value of α for which (3.3.5) and (3.3.6) give equal values of r must be $(1/2,1)$ such values of α are obtained by eliminating r between (3.3.6) and (3.3.5) at $\beta = 0$. Finally we get following result, if

$$\alpha_o = \frac{x_o^2 - 4}{4x_o - 4} \quad \text{where } x_o \text{ is the unique root of}$$

$$x^4 - 4x^3 + 2x^2 - 8 = 0,$$

then for $0 \leq \alpha \leq \alpha_o$, we use the formula (3.3.6) and for $\alpha_o \leq \alpha < 1$, we use the formula (3.3.5). Functions given by (3.2.15) and (3.2.16) show that the bounds are sharp.

Remark : Taking $\beta = 0$ in (3.3.5) and (3.3.6) we obtain results of V.A. Zmorovič [39] which are also obtained by V. Singh and R.M. Goel [34]

3.4 Radius of starlikeness for the class H_α

Theorem 2. Let $f(z) \in H_\alpha$. Let α_o be the unique positive root of the equation $4x^3 - 4x^2 + 5x - 4 = 0$, then $f(z)$ is starlike of order β, $0 \leq \beta < 1$, in

$$0 < |z| < \left[\frac{\frac{\beta}{2}(1+\frac{\beta}{2}) + \alpha - 1 + \sqrt{(\frac{\beta}{2}(1+\frac{\beta}{2})+\alpha-1)^2 + (2-(\frac{1+\beta}{2})^2(2\alpha+\frac{\beta^2}{4})}}{2\alpha + \frac{\beta^2}{4}}\right]^{1/2}$$

if $0 \leq \alpha \leq \alpha_o$ and $\alpha = \beta \neq 0$.

$0 < |z| < 1/\sqrt{2}$ if $\alpha = \beta = 0$,

$$0 < |z| < \frac{1-\beta}{\alpha - \frac{\beta}{2}(1+\alpha) + \sqrt{(1-\alpha)\{\alpha - \alpha\beta + \frac{\beta^2}{4}(1-\alpha)\}}}$$

if $\alpha_o \leq \alpha < 1$.

<u>Proof:</u> Since $f(z) \in H_\alpha$ from lemma 3 we get

$$(3.4.1) \quad -\mathrm{Re}\,\{\frac{zf'(z)}{f(z)} + \beta\} \geq \frac{1 - 2\alpha r + \alpha(2\alpha-1)\, r^2}{(1-r)\ (1-\alpha r)} - \beta$$

$$= \frac{1-\beta-2(\alpha - \frac{\beta}{2}\,(1+\alpha))r + \alpha(2\alpha-\beta-1)r^2}{(1-r)(1-\alpha r)}$$

if $u_o \geq u_1$

$$(3.4.2) \quad -\mathrm{Re}\,\{\frac{zf'(z)}{f(z)} + \beta\} \geq -\frac{2}{1-r^2} + 2\sqrt{\frac{2(1+\alpha r^2)}{1-r^2}} - \frac{2\beta}{2}$$

$$= 2(-\frac{1 + \frac{\beta}{2} - \frac{\beta}{2}\, r^2}{1-r^2} + \sqrt{\frac{2(1+\alpha r^2)}{1-r^2}})$$

if $u_o \leq u_1$

where $u_o = \frac{1}{1-\alpha}\,\{\sqrt{\frac{1+\alpha r^2}{2(1-r^2)}} - \alpha\}$ and $u_1 = \frac{1}{1-r}$. Therefore

$-\mathrm{Re}\,\{\frac{zf'(z)}{f(z)} + \beta\} \geq 0$ if

$$(1-\beta) - 2(\alpha - \frac{\beta}{2}\,(1+\alpha))\ r + \alpha(2\alpha-\beta-1)r^2 \geq 0$$

and

$$-(2\alpha + \frac{\beta^2}{4})\ r^4 - 2((1-\alpha) - \frac{\beta}{2}\,(1 + \frac{\beta}{2}))\ r^2 + 2 - (1 + \frac{\beta}{2})^2 \geq 0.$$

Therefore radii of starlikeness for the class H_α are given

$$(3.4.3) \quad r = \frac{1-\beta}{\alpha - \frac{\beta}{2}\,(1+\alpha) + \sqrt{(1-\alpha)\ \{\alpha-\alpha\beta + \frac{\beta^2}{4}\,(1-\alpha)\}}}$$

$$(3.4.4) \quad r = [\frac{\alpha + \frac{\beta}{2}\,(1 + \frac{\beta}{2}) - 1\sqrt{\{\alpha + \frac{\beta}{2}(1 + \frac{\beta}{2})-1\}^2 + \{2-(1 + \frac{\beta}{2})^2\}(2\alpha + \frac{\beta^2}{4})}}{2\alpha + \frac{\beta^2}{4}}]^{1/2}$$

$$\alpha = \beta \neq 0$$

if $\alpha = \beta = 0$ then

$$r = \frac{1}{\sqrt{2}}$$

The values of α for which (3.4.3) and (3.4.4) give equal values of r must be in (1/2,1). Such values of α are obtained by eliminating r between $u_0 = u_1$ and (3.4.3) at $\beta = 0$. Finally we have the following result, if α_0 is the unique positive root of the equation.

$$4x^3 - 4x^2 + 5x - 4 = 0,$$

then for $0 \leq \alpha \leq \alpha_0$ radius of starlikeness of $f(z)$ is given by (3.4.4) and if $\alpha_0 \leq \alpha < 1$ then radius of starlikeness of $f(z)$ is given by (3.4.3).

Functions given by (3.2.15) and (3.2.19) show that these results are sharp.

3.5 Radius of starlikeness for the class T.

Notations :let

$$\mu(\alpha\ ,\ r) = \frac{1 + (2\alpha - 1)r}{1 + r}$$

$$\eta(\beta\ ,r) = \frac{2r(1 - \beta)}{(1-r)\ (1-(2\beta-1)r)} \quad .$$

Theorem 3. Let $f(z) \in T$, then for $|z| = r$, $0 \leq r < 1$, $f(z)$ is starlike in $0 < |z| < r_0$ where r_0 is the least positive root of the equation

$$\mu(\alpha, r) - \eta(\beta, r) - \eta(\gamma, r) = 0$$

This result is sharp.

Proof : Since $f(z) \in T$ we have

$$(3.5.1) \qquad f(z) = g(z)\, p(z)\, q(z),$$

where $g(z) \in \Sigma^*(\alpha)$, $p(z) \in P(\beta)$ and $q(z) \in P(\gamma)$.

Diff. (3.5.1) w.r.t z we have

$$(3.5.2) \qquad -\frac{zf'(z)}{f(z)} = -\frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)} - \frac{zq'(z)}{q(z)}$$

Therefore

$$(3.5.3) \qquad -\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \frac{1+(2\alpha-1)r}{1+r} - \frac{2r(1-\beta)}{(1-r)\,(1-(2\beta-1)\,r)} - \frac{2r(1-\gamma)}{(1-r)(1-(2\gamma-1)r)}$$

$$= \mu(\alpha,r) - \eta(\beta,r) - \eta(\gamma,r)$$

(3.5.3) has been obtained by using lemma 2 and lemma 4 of chapter 2 in (3.5.2).

Since $\mu(\alpha,0) - \eta(\beta,0) - \eta(\gamma,0) = 1$

and $\lim\limits_{r \to 1} \mu(\alpha,r) - \eta(\beta,r) - \eta(\gamma,r) = -\infty$,

Therefore equation $\mu(\alpha,r) - \eta(\beta,r) - \eta(\gamma,r) = 0$

has always a root in $(0,1)$.

The sharpness of the result follows from the sharpness of lemma 2 and 4 of the chapter 2.

Remark : Taking $\gamma = 1$ in (3.5.3) we get the Theorem 1 of K.S. Padmanabhan [27].

CHAPTER 4

EFFECT OF SECOND COEFFICIENT ON THE RADII OF CONVEXITY AND STARLIKENESS

4.1. In this chapter we shall investigate the effect of second coefficient on the radii of convexity and starlikeness of certain analytic function in D .

Let

$$(4.1.1) \qquad p(z) = 1 + 2az + a_2 z^2 + \ldots$$

where $p(z)$ is regular, $\operatorname{Re}\{p(z)\} > \alpha$, $0 \leq \alpha < 1$, in D and $|a| < (1-\alpha)$. Let $P_{2a}(\alpha)$ denote the class of functions of the form (1). Recently R.S. Gupta [13] investigated the effect of 'a' on the radius of starlikeness of

$$(4.1.2) \qquad f(z) = p(z) - 1 = 2az + a_2 z^2 + \ldots$$

when $p(z) \in P_{2a}(0)$ and $|a| < 1$.

The proof given by R.S. Gupta [13] is lengthy and complicated. In the third section of this chapter we have investigated the effect of 'a' on the radius of starlikeness of $f(z)$ given by (4.1.2) where $p(z) \in P_{2a}(\alpha)$ and $|a| < (1-\alpha)$. It is interesting to note that the proof given here is shorter and simpler than the proof given by R.S. Gupta [13]

In the fourth section of this chapter we have investigated the effect of 'a' on the radius of convexity of

(4.1.3) $$F(z) = z + 2az^2 + a_3 z^3 + \dots$$

when $F(z) \in S^*$ and $|a| < 1$

In the fifth and last section of this chapter we have obtained the radius of starlikeness of

(4.1.4) $$f(z) = \frac{1}{2} \, [zF(z)]' ,$$

where $F(z)$ is given by (4.1.3), in terms of 'a'.

4.2. Before proving the actual results we need the following lemma.

Lemma 1 : If $f(z)$ is regular in $|z| < 1$ and $|f(z)| \leq 1$ there, then

(4.2.1) $$\frac{|f(0)| - |z|}{1 - |f(0)|\,|z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)|\,|z|}$$

The proof of above lemma can be found in [25 , 167] .

4.3. Radius of starlikeness for the class $P_{2a}(\alpha)$.

Theorem 1 : Let $p(z) \in P_{2a}(\alpha)$ and $f(z)$ is given by (4.1.2), then $f(z)$ is starlike for $|z| < r_o$, where

$$r_o = \frac{|a|}{(1-\alpha) + \sqrt{(1-\alpha)^2 - |a|^2}}$$

Proof : Since $\mathrm{Re}\,\{p(z)\} > \alpha$, $0 \leq \alpha < 1$, for $z \in D$, we can write

(4.3.1) $$p(z) = \frac{1 + (1-2\alpha)\, z\phi(z)}{1 - z\,\phi(z)},$$

where $\phi(z)$ is regular in D and $|\phi(z)| \leq 1$ there.

Therefore

(4.3.2) $$z\,\phi(z) = \frac{a}{1-\alpha}\, z + \dots$$

Applying (4.2.1) to $z\,\phi(z)$ we atonce get

$$(4.3.3)\qquad \frac{|z|(|a|-(1-\alpha)|z|)}{1-\alpha-|a|\;|z|} \le |z\phi(z)| \le \frac{|z|(|a|-(1-\alpha)|z|}{1-\alpha+|a|\;|z|}$$

Now from (4.1.2) and (4.3.1), we obtain

$$(4.3.4)\qquad \mathrm{Re}\,\{\frac{zf'(z)}{f(z)}\} = \mathrm{Re}\,\{\frac{zp'(z)}{p(z)-1}\} = \mathrm{Re}\,\{\frac{z^2\phi'(z) + z\phi(z)}{z\phi(z)\;(1-z\phi(z))}\}$$

By using (2.2.5) in (4.3.4) we have

$$(4.3.5)\qquad \{\frac{zf'(z)}{f(z)}\} \ge \mathrm{Re}\,\{\frac{1}{1-z\phi(z)}\} - \frac{|z|^2 - |z\phi(z)|^2}{(1-|z|^2)|z\phi(z)|\;(1-|z\phi(z))|}$$

Let

$$(4.3.6)\qquad |z| = r$$

and

$$(4.3.7)\qquad \frac{1}{1-z\,\phi(z)} = u + iv.$$

Putting values of $|z|$ and $\frac{1}{1-z\,\phi(z)}$ in (4.3.5) we get

$$(4.3.8)\qquad \mathrm{Re}\,\{\frac{zp'(z)}{p(z)-1}\} \ge u - \frac{r^2(u^2+v^2)-(u-1)^2-v^2}{\sqrt{(u-1)^2+v^2}\;(1-r^2)} \equiv H(u,v,r) \quad \text{(say).}$$

Diff. H partially w.r.t. v, we get

$$(4.3.9)\qquad \frac{\partial H}{\partial v} = v\,[-\frac{2}{\sqrt{(u-1)^2+v^2}} + \frac{r^2(u^2+v^2)-(u-1)^2-v^2}{(1-r^2)\;\{(u-1)^2+v^2\}^{3/2}}]$$

Since $|z\,\phi(z)| \le r$ and $|z\,\phi(z)|^2 = \frac{(u-1)^2+v^2}{u^2+v^2}$, we can easily conclude

$$(4.3.10)\qquad r^2\,(u^2+v^2) - (u-1)^2 - v^2 \ge 0\,.$$

In view of (4.3.10), quantity within square brackets of (4.3.S) turms out strictly positive, therefore minimum of H occurs at $v = 0$. Putting $v = 0$ in (4.3.8), we obtain

(4.3.11) $$h(u,r) \equiv H(u,0,r) = u - \frac{r^2u^2-(u-1)^2}{(1-r^2)\,(u-1)}\,.$$ Also at $v = 0$.

(4.3.12) $$|z\,\phi(z)| = \frac{|u-1|}{u}\,.$$

From (4.3.3) and (4.3.12) we have

(4.3.13) $$\frac{1-\alpha+|a|r}{1-\alpha+2|a|r+(1-\alpha)r^2} \le u \le \frac{1-\alpha-|a|r}{(1-\alpha)(1-r^2)}$$

if $1 - u \ge 0$,

and

(4.3.14) $$\frac{1-\alpha-|a|r}{1-\alpha-2|a|r+(1-\alpha)r^2} \le u \le \frac{1-\alpha+|a|r}{(1-\alpha)(1-r^2)}$$

if $(u-1) \ge 0$.

One can easily check that h is a monotone decreasing function of u if $1-u \ge 0$ and it is monotone increasing function of u if $(u-1) \ge 0$. Therefore if $(1-u) \ge 0$, minimum of h occurs at

$$u = u_1 = \frac{1-\alpha-|a|r}{(1-\alpha)\,(1-r^2)}$$

and is equal to

$$\frac{|a| - 2(1-\alpha)\,r+|a|r^2}{(1-r^2)\,(|a|-(1-\alpha)r)} \equiv h\,(u_1,r)\,.$$

If $u-1 \ge 0$, then minimum of h occurs at

$$u = u_2 = \frac{1-\alpha-|a|r}{1-\alpha-2\,|a|\,r+(1-\alpha)\,r^2}$$

and is equal to

$$h(u_2,r) \equiv \frac{(1-\alpha)(|a|-2(1-\alpha)r+|a|r^2)}{(|a|-(1-\alpha)r)(1-\alpha-2|a|r+(1-\alpha)r^2)}$$

It is easy to verify that

$$h(u_1,r) \leq h(u_2,r) \quad \text{for} \quad r \leq \frac{|a|}{1-\alpha} .$$

Therefore absolute minimum of h in $(0, \infty)$ will occur at $u = u_1$.

Hence

$$(4.3.15) \quad \operatorname{Re}\left\{\frac{zp'(z)}{p(z)-1}\right\} \geq \frac{|a|-2(1-\alpha)r+|a|r^2}{(1-r^2)(|a|-1-\alpha\ r)} \text{ provided } r \leq \frac{|a|}{1-\alpha}$$

Therefore

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)-1}\right\} \geq 0$$

for

$$|z| < \frac{|a|}{1-\alpha+\sqrt{(1-\alpha)^2-|a|^2}} \leq \frac{|a|}{1-\alpha}$$

The equality sign in (4.3.15) is attained for the function

$$p(z) = \frac{1-2az+(1-2\alpha)z^2}{1-z^2} .$$

<u>Remark</u> : For $\alpha = 0$, we get the result of R.S. Gupta [13]

4.4. <u>Radius of convexity for the class</u> S^*.

<u>Theorem</u> 2 : <u>If</u> $f(z) = z + 2az^2 + a_3z^3 + \ldots \in S^*$, <u>then the radius of convexity of</u> $f(z)$ <u>is given by the smallest positive root of the equation.</u>

$$(4.4.1) \qquad 1 - 2|a|r - 6r^2 - 2|a|r^3 + r^4 = 0.$$

<u>This result is sharp</u>.

<u>Proof</u> : Since $f(z) \in S^*$, there exists a function $w(z) = z\,\phi(z)$ satisfying Schwarz's lemma such that

$$(4.4.2) \qquad \frac{zf'(z)}{f(z)} = \frac{1+z\phi(z)}{1-z\phi(z)}, \quad z \in D .$$

where $\phi(z)$ satisfies (from (4.3.3) at $\alpha = 0$)

$$(4.4.3) \qquad \frac{|z|(|a|-|z|)}{1-|a|\ |z|} \le |z\phi(z)| \le \frac{|z|(|a|+|z|)}{1+|a|\ |z|}$$

Diff. (4.4.2) logarithmically w.r.t. z then combining it with (4.4.2) we obtain

$$1 + \frac{zf''(z)}{f'(z)} = -1 + \frac{3}{1-z\phi(z)} - \frac{1}{1+z\phi(z)} + \frac{2z^2\phi'(z)}{1-(z\phi(z))^2}$$

Therefore

$$(4.4.4) \quad \mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\} \ge -1+3\mathrm{Re}\{\frac{1}{1-z\phi(z)}\}-\mathrm{Re}\{\frac{1}{1+z\phi(z)}\}-\frac{2(|z|^2-|z\phi(z)|^2)}{(1-|z|^2)|1-(z\phi(z))^2|}$$

The above inequality has been obtained by using (2.2.5).

Let

$$(4.4.5) \qquad |z| = r$$

and

$$(4.4.6) \qquad \frac{1}{1-z\phi(z)} = u + iv \ , \quad \text{then } u \text{ satisfies}$$

$$\left|u - \frac{1}{1-|z|^2}\right| \le \frac{|z|}{1-|z|^2}$$

i.e.

$$(4.4.7) \qquad \frac{1}{1+r} \leq u \leq \frac{1}{1-r}$$

Putting the values of $|z|$ and $\frac{1}{1-z\phi(z)}$ in (4.4.4) we get

$$(4.4.8) \quad \mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\} \geq -1+3u-\frac{2u^2-u+2v^2}{(2u-1)^2+4v^2}-\frac{2\{r^2(u^2+v^2)-(u-1)^2-v^2\}}{(1-r^2)\sqrt{(2u-1)^2+4v^2}}$$

$$\equiv G(u,v,r) \quad \text{(say.)}$$

Diff. $G(u,v,r)$ w.r.t. v we obtain

$$(4.4.9) \quad \frac{\partial G}{\partial v} = 4v \left[-\frac{4u^2-2u+1}{\{(2u-1)^2+4v^2\}^2}+\frac{1}{\sqrt{(2u-1)^2+4v^2}}+\frac{2(r^2(u^2+v^2)-(u-1)^2-v^2)}{(1-r^2)\{(2u-1)^2+4v^2\}^{3/2}}\right]$$

Since $|z\phi(z)| \leq r$ and $|z\phi(z)|^2 = \frac{(u-1)^2+v^2}{u^2+v^2}$, it is easy to conclude that

$$(4.4.10) \qquad r^2(u^2+v^2)-(u-1)^2-v^2 \geq 0.$$

Using (4.4.10) and (4.4.7) it is easy to check that quantity within square brackets of (4.4.9) is strictly positive, therefore minimum of G occurs at $v = 0$. Putting $v = 0$ in (4.4.8) we get

$$(4.4.11) \qquad g(u,r) = G(u,0,r) = -1+3u-\frac{u}{2u-1}-\frac{2(r^2u^2-(u-1)^2)}{(1-r^2)(2u-1)}$$

$$= -\frac{3-r^2}{1-r^2}+4u .$$

Also from (4.3.13) and (4.3.14) at $\alpha = 0$ we have

$$(4.4.12) \qquad \frac{1+|a|r}{1+2|a|r+r^2} \leq u \leq \frac{r(|a|+r)}{1-r^2}$$

Since

$$\frac{r(|a|+r)}{1-r^2} \geq \frac{r(|a|-r)}{1-r^2}$$

$$\frac{1+|a|r}{1+2|a|r+r^2} \leq \frac{1-|a|r}{1-2|a|r+r^2}$$

It is clear from (4.4.11) that g is a monotone increasing function of u, therefore its absolute minimum in $(0, \infty)$ will occur at

$$u = u_1 = \frac{1+|a|r}{1+2|a|r+r^2}$$

and is equal to

$$g(u_1,r) \equiv \frac{1-2|a|r-6r^2-2|a|r^3+r^4}{(1-r^2)(1+2|a|r+r^2)} \,. \tag{4.4.13}$$

Therefore

$$\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \frac{1-2|a|r-6r^2-2|a|r^3+r^4}{(1-r^2)(1+2|a|r+r^2)} \,. \tag{4.4.14}$$

Hence $f(z)$ is convex for $|z| < r_o$ where r_o is the smallest positive root of the equation

$$1-2|a|r-6r^2-2|a|r^3+r^4 = 0 \,.$$

The equality sign in (4.4.14) is attained only for the functions of the form

$$f(z) = \int_0^z \frac{(1-t^2)\,dt}{(1+2at+t^2)^2} \tag{4.4.15}$$

at $z = r$.

<u>Remark</u> : The above result has also been obtained by Tepper [36] by using different technique. But Tepper has not produced an example to show the sharpness of the result.

4.5. We devote this section just to obtain the region of starlikeness of the functions represented by (4.1.4) in terms of their second coefficients.

Theorem 3 : If $f(z) = \frac{1}{2}\,[zF(z)]'$, where $F(z) = z + 2az^2 + a_3z^3 + \ldots \in S^*$, then $f(z)$ is starlike for $|z| < \frac{1}{2|a|}$, where $2/3 < |a| < 1$. This result is sharp.

Proof : Since $f(z) \in S^*$, there exists a function $w(z) = z\phi(z)$ satisfying conditions of Schwarz's lemma such that

(4.5.1) $$\frac{zF'(z)}{F(z)} = \frac{1 + z\phi(z)}{1 - z\phi(z)}, \quad z \in D.$$

By direct calculation we get from (4.5.1)

(4.5.2) $$z\phi(z) = az + \ldots$$

Thus applying (4.2.1) to (4.5.2) we can easily get

(4.5.3) $$\frac{|z|(|a| - |z|)}{1 - |a|\,|z|} \leq |z\phi(z)| \leq \frac{|z|(|a| + |z|)}{1 + |a|\,|z|}$$

Again, from (4.5.1) and (4.1.4) we get

$$f(z) = \frac{F(z)}{1 - z\phi(z)}$$

Logarithmic diff. of $f(z)$ w.r.t. z together with (4.5.1) yields

(4.5.4) $$\frac{zf'(z)}{f(z)} = -2 + \frac{3}{1 - z\phi(z)} + \frac{z^2\,\phi'(z)}{1 - z\phi(z)}$$

Therefore

(4.5.5) $$\text{Re}\,\left\{\frac{zf'(z)}{f(z)}\right\} \geq -2 + 3\,\text{Re}\,\left\{\frac{1}{1 - z\phi(z)}\right\} - \frac{|z|^2 - |z\phi(z)|^2}{(1 - |z|^2)\,|(1 - z\phi(z))|}$$

The above inequality has been obtained by using (2.2.5).

Set $|z| = r$ and $\frac{1}{1-z\,\phi(z)} = u + iv$, (4.5.5) reduces to

$$(4.5.6)\quad \mathrm{Re}\,\{\frac{zf'(z)}{f(z)}\} \geq -2+3u - \frac{r^2(u^2+v^2)-(u-1)^2-v^2}{(1-r^2)\ \sqrt{u^2+v^2}} \equiv T(u,v,r), \quad \text{(say.)}$$

Diff. T partially w.r.t. v we obtain

$$(4.5.7)\quad \frac{\partial T}{\partial v} = v\left[\frac{1}{\sqrt{u^2+v^2}} + \frac{r^2(u^2+v^2)-(u-1)^2-v^2}{(1-r^2)(u^2+v^2)^{3/2}}\right]$$

Since $|z\,\phi(z)| \leq r$ and $|z\,\phi(z)|^2 = \frac{(u-1)^2+v^2}{u^2+v^2}$,

therefore

$$(4.5.8)\qquad r^2(u^2+v^2) - (u-1)^2 - v^2 \geq 0 .$$

In view of (4.5.8), quantity within square brackets of (4.5.7) is strictly positive, therefore minimum of T occurs at $v = 0$. Putting $v = 0$ in (4.5.6), we get

$$t(u,r) \equiv T(u,0,r) = -2+3u - \frac{r^2u^2-(u-1)^2}{(1-r^2)\,u} = \frac{-2(2-r^2)}{1-r^2} + 4u + \frac{1}{(1-r^2)u}$$

Also u satisfies (4.4.12).

It is easy to check that t is a monotone increasing function of u for $r < 3/4$. Therefore absolute minimum of t in $(0, \infty)$ is attained at

$$u = u_1 = \frac{1+|a|\,r}{1+2|a|\,r + r^2}$$

and is equal to

$$t(u_1,r) \equiv \frac{2|a|r^5+(4|a|^2-1)r^4-2|a|r^3-4|a|^2\ r^2+1}{(1-r^2)(1+|a|r)(\ 1+2|a|r+r^2)} \quad \text{for } r < \frac{3}{4}$$

Therefore

$$(4.5.11) \quad \text{Re}\ \{\frac{zf'(z)}{f(z)}\} \geq \frac{2|a|r^5+(4|a|^2-1)r^4-2|a|r^3-4|a|^2r^2+1}{(1-r^2)(1+|a|r)(1+2|a|r+r^2)}$$

Thus $\quad \text{Re}\ \{\frac{zf'(z)}{f(z)}\} \geq 0$

if $2\ |a|\ r^5 + (4\ |a|^2-1)\ r^4-2|a|r^3-4|a|^2r^2+1 \geq 0$

i.e.,

$$(1-r^2)\ (\ \frac{1}{2} - |a|r)\ (1 + 2|a|r + r^2) \geq 0$$

Therefore $f(z)$ is starlike for

$$|z| < \frac{1}{2|a|}$$

The result is sharp for the functions of the form

$$F(z) = \frac{z}{(1+az)^2} \ ,$$

at $\quad z = r.$

<u>Remark</u> : In the limiting case we obtain the result of A.E. Livingston [22] from the above result.

CHAPTER 5

EFFECT OF DROPING FIRST (n-1) COEFFICIENTS ON THE RADII OF CONVEXITY AND STARLIKENESS

5.1 Let S denote the class of regular and univalent functions $f(z)$ in $D = \{z : |z| < 1\}$ which are normalised by the conditions $f(0) = 0$, $f'(0) = 1$. For a fixed α, $0 \leq \alpha < 1$, let $C(\alpha)$ denote the subclass of S, consisting of all functions f satisfying the condition

$$(5.1.1) \qquad \text{Re } \{z \frac{f''(z)}{f'(z)} + 1\} > \alpha \text{ for } z \in D.$$

Let $S^*(\alpha)$ denote the subclass of S, consisting of all functions f satisfying the condition

$$(5.1.2) \qquad \text{Re } \{\frac{z f'(z)}{f(z)}\} > \alpha \text{ for } z \in D.$$

Let $K(\alpha, \beta)$ denote the subclass of S formed by all functions f for which there exists some function $g(z) \in C(\beta)$ such that

$$(5.1.3) \qquad \text{Re } \{\frac{f'(z)}{g'(z)}\} > \alpha,\ 0 \leq \alpha < 1,\ 0 \leq \beta < 1;\ \text{for } z \in D$$

Functions in the classes $C(\alpha)$, $S^*(\alpha)$ and $K(\alpha,\beta)$ are known as convex functionz of order α , starlike functions of order α and close-to-convex functions of order α and type β respectively. We shall denote by S_n, $n = 1,2,3,\ldots$, the class of regular and univalent functions in D having Taylor expansion of the form $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$. It is clear that $S = S_1 \supseteq S_2 \supseteq S_3 \ldots \supseteq S_n$.

Let A_n be the class of functions of the form
$f(z) = z + \sum_{k=n}^{\infty} a_{k+1} z^{k+1}$, $n \geq 1$, analytic in D.

In the 3rd section of this chapter we shall show that if $F(z)$ be in $(S^*\beta)_n$, $(C\beta)_n$ or $(K(\alpha,\beta))_n$, then $f(z) = \frac{1}{1+C} z^{1-C} [z^C F(z)]'$ is starlike of order β, convex of order β or close-to-convex of order α and type β for $|z| < r_n^0$ respectively. Also if $F(z) \varepsilon S_n$ and $\mathrm{Re}\,\{F'(z)\} > \beta$ in D, then $f(z) = \frac{1}{1+C} z^{1-C}[z^C F(z)]'$ satisfies $\mathrm{Re}\,\{f'(z)\} > \beta$ in $\{|z| < r_n'\}$. All our results are sharp. For suitable choice of C, β and n results of S.K. Bajpai[3], S. Chandra [41] S.D. Barnadi [4] , A.E. Livingston [22] and K.S. Padamanabhan [26] follow as special cases of the results derived here.

In the 4th section of this chapter we obtain the radius of starlikeness of $f(z) \varepsilon A_n$ under certain conditions imposed on it. All our results are sharp. Theorem (3.1) of Causey and Merkes [7] and theorems 1,3,4 and 6 of Ratti [31] follows as special cases of the results derived here.

In the 5th section of this chapter we obtain the radius of convexity of $f(z) \varepsilon A_n$ under certain conditions imposed on it. All results are sharp. Theorems 2,4. and 6 of Ratti [30] and theorem 4 of Ram Singh [32] follow as special cases of the results derived here by taking n = 1.

5.2 The following lemmas will be used.

Lemma 1. Suppose that $h(z) = 1 + \sum_{k=n}^{\infty} c_k z^k$, $n \geq 1$, is analytic and satisfies $\mathrm{Re}(h(z)) > 0$ for $z \varepsilon D$, then

$$(5.2.1) \qquad \left|\frac{h'(z)}{h(z)}\right| \leq \frac{2n\,|z|^{n-1}}{1-|z|^{2n}}$$

The proof of the lemma can be found in [24].

Lemma 2. *The function* $H(z) = 1 + \sum_{k=n}^{\infty} c_k z^k$, $n \geq 1$, *is regular and satisfies* $\mathrm{Re}\,(H(z)) > \alpha$, $0 \leq \alpha < 1$, *for* $z \in D$ *iff*

$$(5.2.2) \qquad H(z) = \frac{1+(2\alpha-1)\,z^n\phi(z)}{1+z^n\,\phi(z)}$$

where $\phi(z)$ *is a regular function and satisfies* $|\phi(z)| \leq 1$ *for* $z \in D$.

Proof : Setting $h(z) = \frac{H(z)-\alpha}{1-\alpha}$, we note that $h(z)$ is regular and satisfies $\mathrm{Re}(h(z)) > 0$ in D and $h(0) = 1$. Let $w(z) = \frac{1-h(z)}{1+z(z)}$, where $w(z)$ is regular and $|w(z)| < 1$ in D, also $w(z)$ has nth order zero at origin. Hence we can write $w(z) = z^n\,\phi(z)$ where $\phi(z)$ is regular in D. It can be shown by applying Schwarz's lemma that $|\phi(z)| \leq 1$ in D. Expressing $h(z)$ in terms of $w(z)$, we obtain

$$h(z) = \frac{1-w(z)}{1+w(z)}$$

Thus

$$(5.2.3) \qquad H(z) = \alpha + (1-\alpha)\frac{1-w(z)}{1+w(z)} = \frac{1+(2\alpha-1)\,w(z)}{1+w(z)}$$

$$= \frac{1+(2\alpha-1)\,z^n\,\phi(z)}{1+z^n\,\phi(z)}$$

Conversely if $H(z)$ is given by (5.2.3) it is obvious that $H(z)$ is regular and $\mathrm{Re}\,(H(z)) > \alpha$ in D.

Lemma 3. *Let* $H(z)$ *be given by* (5.2.2) *then*

(5.2.4) $$\mathrm{Re}(H(z)) \geq \frac{1 + (2\alpha-1)|z|^n}{1+|z|^n}$$

Proof.

$$H(z) = \frac{1 + (2\alpha-1)\ z^n\ \phi(z)}{1+z^n\ \phi(z)} - \alpha + \alpha$$

$$= \alpha + (1-\alpha)\ \frac{1 - z^n\ \phi(z)}{1 + z^n\ \phi(z)}$$

Therefore

$$\mathrm{Re}\ (H(z)) = \alpha + (1-\alpha)\ \mathrm{Re}\ \left(\frac{1-z^n\phi(z)}{1+z^n\phi(z)}\right)$$

$$= \alpha + (1-\alpha)\ \frac{1-(|z^n\ \phi(z)|^2)}{|1+z^n\ \phi(z)|^2}$$

$$\geq \alpha + (1-\alpha)\ \frac{1-|z|^n}{1+|z|^n} = \frac{1 + (2\alpha-1)\ |z|}{1+|z|^n},$$

Lemma 4. *Suppose that in the disc* $|z| < 1$, *a function* $f(z)$ *is regular and satisfies there the inequality* $|f(z)| < 1$ *and has the representation* $f(z) = c_o + \sum\limits_{k=n}^{\infty} c_k z^k$, *where* $n \geq 1$. *Then, in the* disc $|z| < 1$, *we have*

(5.2.5) $$|f'(z)| \leq \frac{n|z|^{n-1}}{1-|z|^{2n}}\ (1-|f(z)|^2)$$

with equality holding only when

$$f(z) = \varepsilon\ \frac{z^n + a}{1+ \bar{a}\ z^n}, \quad |\varepsilon| = 1,\ |a| < 1$$

Lemma 5 : If $H(z) = 1 + C_n z^n + C_{n+1} z^{n+1} + \dots$ be regular Re $\{H(z)\} > \alpha$ for $z \in D$, then

$$(5.2.6) \qquad |H'(z)| \leq \frac{2n|z|^{n-1} \operatorname{Re}\{H(z)-\alpha\}}{1-|z|^{2n}}$$

Proof . Setting $h(z) = \frac{H(z)-\alpha}{1-\alpha}$, if we substitute $w(z) = \frac{h(z)-1}{h(z)+1}$ in (5.2.5) we obtain (5.2.6).

5.3 Some radii of starlikeness, convexity, close-to-convexity problem.

Theorem 1. Let $F(z) = z + \sum_{k=n}^{\infty} a_{k+1} z^{k+1}$, $n \geq 1$, be in $(S^*(\beta))_n$, $f(z) = \frac{1}{1+C} z^{1-C} [z^C F(z)]'$, $C = 1,2,3,\dots$, then $f(z)$ is starlike of order β for $|z| < r_n^o$, where

$$r_n^o = \begin{cases} \left\{\dfrac{-(n+1-\beta) + \sqrt{(n+1-\beta)^2 + (C+1)(C+2\beta-1)}}{C+2\beta-1}\right\}^{1/n} \\ \text{if } C+2\beta-1 \neq 0, \\ \text{and} \\ \left(\dfrac{1}{n+1}\right)^{\frac{1}{n}} \quad \text{if } C+2\beta-1 = 0. \end{cases}$$

This result is sharp.

Proof. Since $F(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots$ is in $(S^*(\beta))_n$, Re $\{\frac{zF'(z)}{F(z)}\} > \beta$ for $z \in D$. Hence by lemma 2 there exists a function $w(z) = z^n \phi(z)$ with $|w(z)| \leq |z|^n$ and regular for $z \in D$, such that

$$(5.3.1) \qquad \frac{zF'(z)}{F(z)} = \frac{z^C f(z) - C\int_0^z t^{C-1} f(t)\,dt}{\int_0^z t^{C-1} f(t)\,dt}$$

$$= \frac{1+(2\beta-1)w(z)}{1+w(z)}$$

for $z \varepsilon D$. Solving for $f(z)$ we have

$$(5.3.2) \qquad f(z) = \frac{(C+1) - (C+2\beta-1)\ w(z)}{(1 + w(z))\ z^C} [\int_0^z t^{C-1} f(t)\ dt]$$

Differentiating (5.3.2) logarithmically and then using (5.3.1) we obtain

$$(5.3.3) \qquad \frac{zf'(z)}{f(z)} = \frac{z(C+2\beta-1)w'(z)}{(C+1)+(C+2\beta-1)w(z)} - \frac{zw'(z)}{1+w(z)} + \frac{1+(2\beta-1)w(z)}{1+w(z)}$$

$$= - \frac{2\ (1-\beta)\ zw'(z)}{(1+w(z))(C+1+(C+2\beta-1)w(z))} + \frac{1+(2\beta-1)w(z)}{1+w(z)}$$

Therefore

$$(5.3.4) \qquad \frac{zf'(z)}{f(z)} - \beta = (1-\beta)\ [\frac{1-w(z)}{1+w(z)} - \frac{2z\ w'(z)}{(1+w(z))(C+1+(C+2\beta-1)w(z))}]$$

But

$$(5.3.5) \qquad \text{Re}\ \{\frac{1-w(z)}{1+w(z)}\} = \frac{1 - |w(z)|^2}{|1 + w(z)|^2}$$

And

$$(5.3.6) \qquad \text{Re}\ \{\frac{2z\ w'(z)}{(1+w(z))(C+1+(C+2\beta-1)w(z))}\}$$

$$\leq \frac{2\ |z|\ |w'(z)|}{|(1+w(z))|\ |(C+1+(C+2\beta-1)w(z))|}$$

$$\leq \frac{2n\ |z|^n\ (1-|w(z)|^2)}{(1-|z|^{2n})|(1+w(z))|\ |(C+1+(C+2\beta-1)w(z))|}$$

The last inequality has been obtained by using (5.2.5).

Thus from (5.3.4) we note that $f(z)$ is starlike of order β, if

$$\frac{2n\,|z|^n\,(1-|w(z)|^2)}{(1-|z|^{2n})\,|(1+w(z))|\,\,|(C+1+(C+2\beta-1)w(z))|} \le \frac{1-|w(z)|^2}{|1+w(z)|^2}$$

or

$$(5.3.7)\qquad \frac{2n|z|^n}{1-|z|^{2n}} \le \left|\frac{C+1+(C+2\beta-1)w(z)}{1+w(z)}\right| = (C+1)\left|\frac{1+\frac{(C+2\beta-1)}{C+1}w(z)}{1+w(z)}\right|$$

Since $|w(z)| \le |z|^n = r^n < 1$ and $\frac{C+2\beta-1}{C+1} \le 1$, we have

$$(5.3.8)\qquad \left|\frac{1+\frac{(C+2\beta-1)}{C+1}}{1+w(z)}\right| \ge \left|\frac{1+\frac{C+2\beta-1}{C+1}|z|^n}{1+|z|^n}\right|$$

Hence, by (5.3.7) and (5.3.8) we obtain that $f(z)$ is in $(S^*(\beta))_n$ if,

$$\frac{2n|z|^n}{1-|z|^{2n}} \le \frac{C+1+(C+2\beta-1)|z|^n}{1+|z|^n}$$

or

$$(5.3.9)\qquad \frac{2n|z|^n}{1-|z|^n} \le (C+1)+(C+2\beta-1)\,|z|^n$$

From (5.3.9) we have

$$(1+C)-2(n+1-\beta)\,|z|^n-(C+2\beta-1)\,|z|^{2n} \ge 0$$

Let

$$(5.3.10) \qquad P(r) \equiv (1+C) - 2(n+1-\beta)\, r^n - (C+2\beta-1)\, r^{2n}$$

Thus $f(z)$ will be starlike of order β for $|z| < r_n^o$, where r_n^o is the least positive root of the polynomial (5.3.10) given by

$$r_n^o = \left\{\frac{-(n+1-\beta) + \sqrt{(n+1-\beta)^2 + (C+1)(C+2\beta-1)}}{C+2\beta-1}\right\}^{1/n}$$

To see that the result is sharp for each C and n, consider the function

$F(z) = \dfrac{z}{(1-z^n)^{\frac{2}{n}(1-\beta)}} \in (S^*(\beta))_n, 0 \leq \beta < 1.$ For this function, we have

$$f(z) = \frac{1}{1+C}\, z^{1-C} \left[\frac{z^{C+1}}{(1-z^n)^{(2/n)(1-\beta)}}\right]'$$

$$= \frac{1}{C+1}\left[\frac{z(1+C) - (C+2\beta-1)\, z^{n+1}}{(1-z^n)^{(2/n)(1-\beta)+1}}\right]$$

By direct computation we obtain

$$\frac{zf'(z)}{f(z)} - \beta = (1-\beta)\left[\frac{(C+1)+2(n+1-\beta)\, z^n - (C+2\beta-1)\, z^{2n}}{(1-z^n)(C+1-(C+2\beta-1)\, z^n)}\right].$$

Thus

$$\frac{zf'(z)}{f(z)} - \beta = 0 \quad \text{for } z^n = -(r_n^o)^n$$

Hence $f(z)$ is not starlike in any circle $|z| < r$ if $r > r_n^o$.

<u>Theorem</u> 2. Let $F(z) = z + \sum_{k=n}^{\infty} a_{k+1} z^{k+1}$, $n \geq 1$, be in $(C(\beta))_n$, $f(z) = \frac{1}{1+C}\, z^{1-C}\, [z^C F(z)]'$, $C = 1,2,3, \ldots$, then $f(z)$ is convex

of order β *for* $|z| < r_n^o$, *where*

$$r_n^o = \begin{cases} \left\{\dfrac{-(n+1-\beta) + \sqrt{(n+1-\beta)^2 + (C+1)(C+2\beta-1)}}{C+2\beta-1}\right\}^{1/n} \\ \text{if } C+2\beta-1 \neq 0, \\ \text{and} \\ \left(\dfrac{1}{n+1}\right)^{\frac{1}{n}} \text{ if } C+2\beta-1 = 0 \end{cases}$$

This result is sharp.

Proof. We have

$$(1+C)\, f'(z) = (1+C)\, F'(z) + z\, F''(z)$$

or

$$(1+C) \operatorname{Re} \left\{\frac{f'(z)}{F'(z)}\right\} = C + \operatorname{Re} \left\{\frac{1+zF''(z)}{F'(z)}\right\}.$$

Since $F(z) \in (C(\beta))_n$, $\operatorname{Re} \left\{1 + \frac{zF''(z)}{F'(z)}\right\} > \beta$ for $z \in D$, therefore it is easy to verify that

$$\operatorname{Re} \left\{\frac{f'(z)}{F'(z)}\right\} > \beta \text{ for } z \in D.$$

Hence $f(z)$ is close-to-convex functions of order β with respect to $F(z)$ in D.

To show that $f(z)$ is convex of order β for $|z| < r_n^o$ we have

$$z\, f'(z) = \frac{1}{1+C}\, z^{1-C} \left[z^C\, (zF'(z))\right]'$$

Since $F(z) \in (C(\beta))_n$, $z\, F'(z) \in (S^*(\beta))_n$; hence $zf'(z)$ is starlike of order β for $|z| < r_n^o$ by theorem 1. Therefore $f(z)$ is convex

of order β for $|z| < r_n^o$.

To show that the result is sharp for each C and n, consider the function

$$F(z) = \int_0^z \frac{d\sigma}{(1-\sigma^n)^{(2/n)(1-\beta)}} \in (C(\beta))_n \text{ as}$$

$$z\, F'(z) = \frac{z}{(1-z^n)^{(2/n)(1-\beta)}} \in (S^*(\beta))_n.$$

For this function

$$f(z) = \frac{1}{1+C} z^{1-C} \left[z^C \int_0^z \frac{d\sigma}{(1-\sigma^n)^{(2/n)(1-\beta)}}\right]'$$

$$= \frac{1}{1+C} \left[\frac{z}{(1-z^n)^{(2/n)(1-\beta)}} + C \int_0^z \frac{d\sigma}{(1-\sigma^n)^{(2/n)(1-\beta)}}\right]$$

By direct computation we obtain

$$1 + \frac{zf''(z)}{f'(z)} - \beta = (1-\beta) \left[\frac{(1+C)+2(n+1-\beta)\ z^n - (C+2\beta-1)\, z^{2n}}{(1-z^n)(C+1-(C+2\beta-1)\ z^n}\right]$$

Thus

$$1 + \frac{zf''(z)}{f'(z)} - \beta = 0 \quad \text{for} \quad z^n = -(r_n^o)^n,$$

hence $f(z)$ is not convex of order β for any circle $|z| < r$, if $r > r_n^o$.

<u>Theorem</u> 3. <u>Let</u> $F(z) = z + \sum_{k=n}^{\infty} a_{k+1} z^{k+1}$, $n \geq 1$, <u>be</u> <u>in</u> $(K(\alpha, \beta))_n$, $f(z) = \frac{1}{1+C} z^{1-C} [z^C F(z)]'$, $C = 1,2,3,\ldots$, <u>then</u> $f(z)$ <u>is</u> <u>close-to-convex</u> <u>of</u> <u>order</u> α <u>and</u> <u>type</u> β <u>for</u> $|z| < r_n^o$, <u>where</u>

$$r_n^o = \begin{cases} \left\{\dfrac{-(n+1-\beta) + \sqrt{(n+1-\beta)^2 + (C+1)(C+2\beta-1)}}{C+2\beta-1}\right\}^{1/n} \\ \text{if } C + 2\beta-1 \neq 0, \\ \text{and} \\ \left(\dfrac{1}{n+1}\right)^{\frac{1}{n}} \quad \text{if } C + 2\beta - 1 = 0 \end{cases}$$

<u>This result is sharp</u>.

<u>Proof</u>. Since $F(z) \in (K(\alpha,\beta))_n$, therefore there exists a function $G(z) \in (S^*(\beta))_n$ such that for $z \in D$

$$\text{Re}\left\{\frac{zF'(z)}{G(z)}\right\} > \alpha \text{ and } \text{Re}\left\{\frac{zG'(z)}{G(z)}\right\} > \beta .$$

Further, since $G(z) \in (S^*(\beta))_n$, from theorem 1, we have

$$\text{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > \beta \text{ for } |z| < r_n^o ,$$

where

$$g(z) = \frac{1}{1+C} z^{1-C} [z^C G(z)]'$$

Therefore

$$(5.3.11) \qquad \frac{zF'(z)}{G(z)} = \frac{z^C f(z) - C\int_0^z t^{C-1} f(t)\, dt}{\int_0^z t^{C-1} g(t)\, dt}$$

Let

$$\frac{zF'(z)}{G(z)} = P(z), \text{ where } P(z) \text{ is regular, } P(0) = 1$$

and $\text{Re}\{P(z)\} > \alpha$ for $z \in D$.

Thus from (5.3.11) we have

$$z^C f(z) = C \int_0^z f(t)\, t^{C-1}\, dt + P(z) \int_0^z t^{C-1}\, g(t)\, dt. \tag{5.3.12}$$

Differentiating (5.3.12) with respect to z and then deviding by g(z) throughout, we obtain

$$\frac{zf'(z)}{g(z)} = P(z) + P'(z)\, \frac{z^{1-C} \int_0^z t^{C-1}\, g(t)\, dt}{g(z)}$$

Therefore

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)} - \alpha\right\} \geq \operatorname{Re}\{P(z) - \alpha\} - |P'(z)| \left|\frac{z^{1-C}\int_0^z t^{C-1} g(t)\, dt}{g(z)}\right| \tag{5.3.13}$$

But

$$\frac{1}{z^C\, g(z)} \int_0^z t^{C-1}\, g(t)\, dt = \frac{G(z)}{C\, G(z) + z\, G'(z)} \tag{5.3.14}$$

Since $G(z) \in (S^*(\beta))_n$ we have

$$\frac{zG'(z)}{G(z)} = \frac{1+(2\beta-1)w(z)}{1+w(z)} \tag{5.3.15}$$

Therefore we have

$$\left\{C + \frac{zG'(z)}{G(z)}\right\}^{-1} = \left\{C + \frac{1+(2\beta-1)w(z)}{1+w(z)}\right\}^{-1} = \left|\frac{C+1 + (C+2\beta-1)\, w(z)}{1 + w(z)}\right|^{-1} \tag{5.3.16}$$

Thus using (5.3.16) in (5.3.13) we obtain

$$(5.3.12) \qquad z^C f(z) = C \int_0^z f(t)\, t^{C-1}\, dt + P(z) \int_0^z t^{C-1}\, g(t)\, dt.$$

Differentiating (5.3.12) with respect to z and then deviding by $g(z)$ throughout, we obtain

$$\frac{zf'(z)}{g(z)} = P(z) + P'(z)\, \frac{z^{1-C} \int_0^z t^{C-1}\, g(t)\, dt}{g(z)}$$

Therefore

$$(5.3.13) \qquad \mathrm{Re}\, \{\frac{zf'(z)}{g(z)} - \alpha\} \geq \mathrm{Re}\, \{P(z) - \alpha\} - |P'(z)|\, |\frac{z^{1-C}\int_0^z t^{C-1} g(t)\, dt}{g(z)}$$

But

$$(5.3.14) \qquad \frac{1}{z^C\, g(z)} \int_0^z t^{C-1}\, g(t)\, dt = \frac{G(z)}{C\, G(z) + z\, G'(z)}$$

Since $G(z) \varepsilon (S^*(\rho))_n$ we have

$$(5.3.15) \qquad \frac{zG'(z)}{G(z)} = \frac{1+(2\beta-1)w(z)}{1+w(z)}$$

Therefore we have

$$(5.3.16) \quad \{C + \frac{zG'(z)}{G(z)}\}^{-1} = \{C + \frac{1+(2\beta-1)w(z)}{1+w(z)}\}^{-1} = |\frac{C+1 + (C+2\beta-1)\, w(z)}{1 + w(z)}|^{-1}$$

Thus using (5.3.16) in (5.3.13) we obtain

(5.3.12) $$z^C f(z) = C \int_0^z f(t)\, t^{C-1}\, dt + P(z) \int_0^z t^{C-1} g(t)\, dt.$$

Differentiating (5.3.12) with respect to z and then deviding by $g(z)$ throughout, we obtain

$$\frac{zf'(z)}{g(z)} = P(z) + P'(z) \frac{z^{1-C} \int_0^z t^{C-1} g(t)\, dt}{g(z)}$$

Therefore

(5.3.13) $$\text{Re}\left\{\frac{zf'(z)}{g(z)} - \alpha\right\} \geq \text{Re}\,\{P(z) - \alpha\} - |P'(z)|\, \left|\frac{z^{1-C}\int_0^z t^{C-1} g(t)\, d}{g(z)}\right.$$

But

(5.3.14) $$\frac{1}{z^C\, g(z)} \int_0^z t^{C-1}\, g(t)\, dt = \frac{G(z)}{C\, G(z) + z\, G'(z)}$$

Since $G(z) \in (S^*(\beta))_n$ we have

(5.3.15) $$\frac{zG'(z)}{G(z)} = \frac{1+(2\beta-1)w(z)}{1+w(z)}$$

Therefore we have

(5.3.16) $$\left\{C + \frac{zG'(z)}{G(z)}\right\}^{-1} = \left\{C + \frac{1+(2\beta-1)w(z)}{1+w(z)}\right\}^{-1} = \left|\frac{C+1 + (C+2\beta-1)\, w(z)}{1 + w(z)}\right|^{-1}$$

Thus using (5.3.16) in (5.3.13) we obtain

$$\text{(5.3.17)} \qquad \text{Re}\left\{\frac{zf'(z)}{g(z)} - \alpha\right\} \geq \left[\text{Re}\{P(z) - \alpha\} - |zP'(z)| \; \left|\frac{1 + w(z)}{C+1+(C+2\beta-1)w(z)}\right|\right]$$

$$\geq \text{Re}\{P(z) - \alpha\}\left\{1 - \frac{2n|z|^n(1+|z|^n)}{(1-|z|^{2n})(C+1+(C+2\beta-1)|z|^n}\right\}$$

$$= \text{Re}\{P(z) - \alpha\}\left\{\frac{(C+1) - 2(n+1-\beta)|z|^n - (C+2\beta-1)|z|^n}{(1-|z|^n)(C+1+(C+2\beta-1)|z|^n}\right\}$$

The last inequality has been obtained by using lemma 5 and (5.3.8)

Therefore $f(z) \in (K(\alpha,\beta))_n$, if $|z| < r_n^o$

To show that the result is sharp, consider $F(z) = G(z)$

$= \dfrac{z}{(1-z^n)^{(2/n)(1-\beta)}} \in (S^*(\beta))_n$, therefore to $(K(\alpha,\beta))_n$, where $\alpha = \beta$.

__Theorem__ 4. __Let__ $F(z) = z + \sum_{k=n}^{\infty} a_{k+1} z^{k+1}, n \geq 1$, __be regular and have the property__ $\text{Re}\{F'(z)\} > \beta$ __for__ $z \in D$, $f(z) = \frac{1}{1-C} z^{1-C} [z^C F(z)]'$, $C = 1,2,3,\ldots$ __then__ $\text{Re}\{f'(z)\} > \beta$ __for__ $|z| < r_n'$, __where__

$$r_n' = \left\{\frac{-n + \sqrt{n^2 + (C+1)^2}}{C+1}\right\}^{1/n}.$$

__This result is sharp.__

__Proof.__ Let $F'(z) = P(z)$, where $P(0) = 1$, and $\text{Re}\{P(z)\} > \beta$ for $z \in D$.
Then we have

$$(1+C)\, f'(z) = z\, F''(z) + (1+C)\, F'(z) = z\, P'(z) + (1+C)\, P(z).$$

Therefore

$$(5.3.18) \qquad (1+C)\,\mathrm{Re}\,(f'(z)-\beta)$$

$$\geq (1+C)\,\mathrm{Re}\,\{P(z)-\beta\} - |z\ \ P''(z)|$$

$$\geq \mathrm{Re}\,\{P(z)-\beta\}\,\{1 + C - \frac{2n\,|z|^{n}}{1-|z|^{2n}}\}$$

$$= \mathrm{Re}\,\{P(z)-\beta\}\,[\frac{(1-C)-2n\,|z|^{n} - (C+1)\,|z|^{2n}}{1-|z|^{2n}}]$$

The last inequality has been obtained by using lemma 5.

Therefore $\mathrm{Re}\,f'(z) > \beta$ for $|z| < r_n'$, where r_n' is the least positive root of the polynomial

$$(5.3.19) \qquad (1+C) - 2n\,r^{n} - (1+C)\,r^{2n} = 0.$$

To show that the result is sharp, consider

$$F(z) = \int_0^z \frac{1-(2\beta-1)\,\sigma^n}{1-\sigma^n}\,d\sigma$$

It is clear that $\mathrm{Re}\,\{F'(z)\} > \beta$.

Therefore

$$f(z) = \frac{1}{1+C}\,z^{1-C}\,[z^C \int_0^z \frac{1-(2\beta-1)\sigma^n}{1-\sigma^n}\,d\sigma]' = \frac{1}{1+C}\,[\frac{z(1-(2\beta-1)\,z^n}{1-z^n}$$

$$+ C\int_0^z \frac{1-(2\beta-1)\sigma^n}{1-\sigma^n}\,d\sigma].$$

By direct computation, we have

$$f'(z) - \beta = \frac{1-\beta}{1+C} \left[\frac{(1+C) + 2n\ z^n - (C+1)\ z^{2n}}{(1-z^n)^2}\right]$$

Thus

$$f'(z) - \beta = 0 \quad \text{for} \quad z^n = -(r_n')^n, \quad \text{hence Re } \{f'(z)\} \nsucc \beta$$

in any circle $|z| < r$, if $r > r_n'$.

5.4 <u>Some radius of starlikeness problems.</u>

<u>Theorem</u> 5. <u>Let</u> $f(z)$ <u>and</u> $g(z)$ <u>be in</u> A_n <u>and</u> Re $\{\frac{g(z)}{s(z)}\} > 0$ <u>for</u> $z \in D$, <u>where</u> $s(z) = z + \sum_{k=n}^{\infty} b_{k+1} z^{k+1}$, $n \geq 1$, <u>is starlike of order</u> α <u>in</u> D. <u>If</u> Re $\{\frac{f(z)}{g(z)}\} > 0$ <u>in</u> D, <u>then</u> $f(z)$ <u>is univalent and starlike in</u> $|z| < r_n(\alpha)$, <u>where</u>

$$r_n(\alpha) = \left\{\frac{1}{2n+1-\alpha + \sqrt{(\alpha^2 - 4\alpha n + 4n + 4n^2)}}\right\}^{1/n}$$

<u>This result is sharp.</u>

<u>Proof</u> : Clearly $f(z) = s(z)\ h(z)\ k(z)$, where $h(z)$ and $k(z)$ satisfy the conditions of lemma 1.

Therefore

$$\frac{zf'(z)}{f(z)} = \frac{zh'(z)}{h(z)} + \frac{zk'(z)}{k(z)} + \frac{zs'(z)}{s(z)}$$

$$\text{(5.4.1)} \qquad \text{Re } \left\{\frac{zf'(z)}{f(z)}\right\} \geq -\frac{4n|z|^n}{1-|z|^{2n}} + \text{Re } \left\{\frac{zs'(z)}{s(z)}\right\}$$

Since Re $\{\frac{zs'(z)}{s(z)}\} > \alpha$, by lemma 2 we have

$$(5.4.2) \qquad \frac{zs'(z)}{s(z)} = \frac{1 + (2\alpha - 1)\ \omega(z)}{1 + \omega(z)}$$

Hence by lemma 3, we have

$$(5.4.3) \qquad \text{Re } \{\frac{zs'(z)}{s(z)}\} \geq \frac{1 + (2\alpha - 1)|z|^n}{1 + |z|^n} .$$

Again, using (5.4.3) in (5.4.1)

$$(5.4.4) \qquad \text{Re } \{\frac{zf'(z)}{f(z)}\} \geq -\frac{4n|z|^n}{1-|z|^{2n}} + \frac{1+(2\alpha - 1)|z|^n}{1 + |z|^n} .$$

$$= \frac{-4n\ |z|^n + 1 + (2\alpha - 1)\ |z|^n - |z|^n - (2\alpha - 1)\ |z|^{2n}}{1 - |z|^{2n}}$$

$$= \frac{1 - (4n + 2 - 2\alpha)\ |z|^n - (2\alpha - 1)\ |z|^{2n}}{1 - |z|^{2n}}$$

It follows from (5.4.4) that $f(z)$ is univalent and starlike for $|z| < r_n(\alpha)$, where $r_n(\alpha)$ is as stated in the theorem.

Let $s(z) = \dfrac{z}{(1 - z^n)^{(\frac{2-2\alpha}{n})}}$, $g(z) = \dfrac{z(1+z^n)}{(1-z^n)^{(\frac{n+2-2\alpha}{n})}}$

and $f(z) = \dfrac{z(1+z^n)^2}{(1-z^n)^{(\frac{2n+2-2\alpha}{n})}}$, then conditions of the theorem are satisfies

and

$$\frac{zf'(z)}{f(z)} = \frac{1 + (4n + 2 - 2\alpha)\, z^n - (2\alpha - 1)\, z^{2n}}{1 - z^{2n}}$$

Thus $\frac{zf'(z)}{f(z)} = 0$ for $z = -r_n(\alpha)$, therefore $f(z)$ is not univalent in any circle $|z| < r$, if $r > r_n(\alpha)$.

Theorem 6 : Let $f(z)$ and $g(z)$ be in A_n and $\operatorname{Re}\{\frac{g(z)}{z}\} > 0$ for $z \in D$. If $|\frac{f(z)}{g(z)} - 1| < 1$ for $z \in D$, then $f(z)$ is starlike and univalent for $|z| < r_n$, where

$$r_n = \left\{\frac{\sqrt{9n^2 + 4n+4} - 3n}{2(n+1)}\right\}^{1/n}$$

This result is sharp.

Proof : Let $\frac{f(z)}{g(z)} - 1 = h(z)$. The hypothesis of the theorem implies that $h(z)$ is analytic for $z \in D$, $h(0) = 0$ and $|h(z)| < 1$ for $z \in D$. We can write $h(z) = z^n\, \phi(z)$ where $\phi(z)$ is analytic and $|\phi(z)| \leq 1$ for $z \in D$.

Therefore, we get

$$f(z) = g(z)\,(1 + z^n\, \phi(z))$$

The logarithmic diff. of $f(z)$ thus yields.

$$(5.4.5) \qquad \frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{z^{n+1}\, \phi'(z) + z^n\, \phi(z)}{1 + z^n\, \phi(z)}$$

Let $p(z) = \frac{g(z)}{z}$, where $p(z)$ satisfies the conditions of lemma 1. Also

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zp'(z)}{p(z)}$$

Therefore

$$(5.4.6) \qquad \text{Re}\ \left\{\frac{zg'(z)}{g(z)}\right\} \geq 1 - \frac{2n|z|^n}{1-|z|^{2n}}\ .$$

Let $\omega(z) = z^n\ \phi(z)$, then we have

$$\frac{z^{n+1}\ \phi'(z) + z^n\ \phi(z)}{1 + z^n\ \phi(z)} = \frac{z\omega'(z)}{1+\omega(z)}$$

But

$$(5.4.7) \qquad \text{Re}\ \left\{\frac{z\omega'(z)}{1+\omega(z)}\right\} \leq \left|\frac{z\omega'(z)}{1+\omega(z)}\right|$$

$$\leq \frac{n|z|^n\ (1-|\omega(z)|^2)}{(1-|z|^{2n})\ (1-|\omega(z)|)}$$

$$\leq \frac{n|z|^n}{1-|z|^n}$$

The last inequality has been obtained by using (5.2.5) .

Thus from (5.4.5), we obtain,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 1 - \frac{2n\ |z|^n}{1-|z|^{2n}} - \frac{n|z|^n}{1-|z|^n}$$

$$= \frac{1-|z|^{2n} - 2n|z|^n - n|z|^n - n|z|^{2n}}{1-|z|^{2n}}$$

$$= \frac{1 - 3n|z|^n - (n+1)\ |z|^{2n}}{1-|z|^{2n}}$$

Thus for $|z| < r_n$, where r_n is as stated in the theorem, $f(z)$ is univalent and starlike.

To show that the result is best possible, we consider

$$f(z) = \frac{z(1+z^n)^2}{1-z^n},\ g(z) = \frac{z(1+z^n)}{1-z^n}$$

It is easy to see that conditions of the theorem are satisfied, but $f(z)$ ceases to be univalent in any circle $|z| < r$, if $r > r_n$.

<u>Theorem</u> 7 : <u>Let</u> $f(z)$ <u>be in</u> A_n <u>and</u> $g(z)$ <u>be in</u> $S_n^*(\alpha)$. <u>If</u> $\operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\} > 0$ <u>for</u> $z \in D$, <u>then</u> $f(z)$ <u>is starlike and univalent for</u> $|z| < r_n'(\alpha)$, <u>where</u>

$$r_n'(\alpha) = \left\{\frac{\sqrt{\alpha^2 - 2n\,\alpha + n^2+2n} - (n+1-\alpha)}{(2\alpha-1)}\right\}^{1/n}$$

<u>if</u> $\alpha \neq \frac{1}{2}$ and

$$r_n'\left(\frac{1}{2}\right) = \left(\frac{1}{2n+1}\right)^{1/n},\ \text{if}\ \alpha = \frac{1}{2}.$$

This result is sharp.

Proof . Let $h(z) = \frac{f(z)}{f(z)}$, where $h(z)$ satisfies the conditions of lemma 1, then

$$\frac{zf'(z)}{f(z)} = \frac{zh'(z)}{h(z)} + \frac{zg'(z)}{f(z)}$$

Using (5.4.3) and lemma 1, we obtain

$$\mathrm{Re}\,\{\frac{zf'(z)}{f(z)}\} \geq \frac{1+(2\alpha-1)|z|^n}{1+|z|^n} - \frac{2n\ |z|^n}{1-|z|^{2n}}$$

$$= \frac{-(2\alpha-1)|z|^{2n} - (2n+2-2\alpha)|z|^n + 1}{1-|z|^{2n}}$$

Thus for $|z| < r_n'(\alpha)$, where $r_n'(\alpha)$ is as stated in the theorem, $f(z)$ is starlike and univalent. To show that the result is sharp consider

$$f(z) = \frac{1+z^n}{(1-z^n)^{(\frac{n+2-2\alpha}{n})}} \quad \text{and} \quad g(z) = \frac{z}{(1-z^n)^{(\frac{2-2\alpha}{n})}}$$

One can easily check that $f(z)$ satisfies the conditions of the theorem but $f(z)$ is not univalent in any circle $|z| < r$, if $r > r_n'(\alpha)$.

Theorem 8. Let $f(z)$ be in A_n and $g(z) = z + \sum_{k=n}^{\infty} b_{k+1} z^{k+1}$, $n \geq 1$, starlike of order α in D. If $\left|\frac{f(z)}{g(z)} - 1\right| < 1$, then $f(z)$ is starlike and univalent for $|z| < r_n^o(\alpha)$, where $r_n^o(\alpha)$ is given by

$$r_n^o(\alpha) = \frac{\sqrt{4\alpha^2 - 4\alpha n + n^2 + 8n} - (n+2-2\alpha)}{2(n + 2\alpha - 1)}\}^{1/n}$$

If $\alpha \neq 0$

$$= \{\frac{\sqrt{n^2 + 8n} - (n+2)}{2(n-1)}\}^{1/n}, \quad \alpha = 0 \text{ and } n \neq 1.$$

This result is sharp.

Proof. Let $h(z) = \frac{f(z)}{g(z)} - 1$, where $h(0) = 0$ and $|h(z)| < 1$ in D therefore by Schwarz's lemma we have $h(z) = z^n \ \phi(z)$, where $\phi(z)$ is regular and $|\phi(z)| < 1$ for $z \in D$. Thus

$$f(z) = g(z)\ (1 + z^n\ \phi(z)).$$

Therefore

$$(5.4.8) \qquad \frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{z^n(z\phi'(z) + \phi(z))}{1 + z^n\ \phi(z)}$$

Using (5.4.3) and (5.4.7) in (5.4.8) we obtain

$$\text{Re}\ \{\frac{zf'(z)}{f(z)}\} \geq \frac{1 + (2\alpha-1)|z|^n}{1 + |z|^n} - \frac{n\ |z|^n}{1-|z|^n}$$

$$= \frac{1+(2\alpha-1)|z|^n - |z|^n - (2\alpha-1)|z|^{2n} - n|z|^n - n|z|^{2n}}{1 - |z|^{2n}}$$

$$= \frac{1 - (n + 2 - 2\alpha)\ |z|^n - (n+2\alpha-1)\ |z|^{2n}}{1 - |z|^{2n}}$$

Thus for $|z| < r_n^o(\alpha)$, where $r_n^o(\alpha)$ is same as stated in the theorem, $f(z)$ is starlike and univalent. To show that the result is sharp, consider

$$f(z) = \frac{z(1+z^n)}{(1-z^n)^{\left(\frac{2-2\alpha}{n}\right)}} \quad \text{and} \quad g(z) = \frac{z}{(1-z^n)^{\left(\frac{2-2\alpha}{n}\right)}}$$

Thus conditions of the theorem are satisfied and

$$\frac{zf'(z)}{f(z)} = \frac{1 + (n+2 - 2\alpha)\, z^n - (n + 2\alpha - 1)\, z^{2n}}{1 - z^{2n}}$$

Thus $f(z)$ is not univalent in any circle $|z| < r$, if $r > r_n^o(\alpha)$.

5.5 Some radius of convexity problems :

Theorem 9. Let $f(z)$ and $g(z)$ be in A_n and $\mathrm{Re}\,(g'(z)) > 0$ in D. If $\mathrm{Re}\,(f'(z)/g'(z)) > 0$ in D, then $f(z)$ is convex and univalent for $|z| < r_n$, where,

$$r_n = \left(\sqrt{(4n^2+1)} - 2n \right)^{1/n}$$

This result is sharp.

Proof. Let $\frac{f'(z)}{g'(z)} = h(z)$, where $h(z)$ satisfies the conditions of lemma 1.

Thus

(5.5.1) $\qquad f'(z) = h(z)\, g'(z)$

The logarithmic diff. of (5.5.1) yields

$$\frac{f''(z)}{f'(z)} = \frac{h'(z)}{h(z)} + \frac{g''(z)}{g'(z)}$$

or

$$(5.5.2) \qquad 1 + \frac{zf''(z)}{f'(z)} = \frac{zh'(z)}{h(z)} + \frac{zg''(z)}{g'(z)} + 1$$

Using lemma 1 in (5.5.2), we obtain

$$\mathrm{Re}\ (1 + \frac{z\ f''(z)}{f'(z)}) \geq 1 - \frac{2n|z|^n}{1-|z|^{2n}} - \frac{2n\ |z|^n}{1-|z|^{2n}}$$

$$= \frac{1 - 4n|z|^n - |z|^{2n}}{1 - |z|^{2n}}$$

Therefore for $|z| < r_n$, where r_n is as stated in the theorem, $f(z)$ is convex and univalent.

To show that the result is sharp, consider

$$f(z) = \int_0^z \left(\frac{1+s^n}{1-s^n} \right)^2 ds$$

and $g(z) = \int_0^z \left(\frac{1+s^n}{1-s^n} \right) ds$. It is easy to verify that the conditions of the theorem are satisfied, but

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{2n\ z^n}{1+z^n} + \frac{2n\ z^n}{1 - z^n}$$

$$= \frac{1+4n\ z^n - z^{2n}}{1 - z^{2n}} = 0$$

at $z^n = -(\sqrt{4n^2+1} - 2n)$

Therefore $f(z)$ is not convex and univalent in any circle $|z| < r$, if $r > r_n$.

<u>Theorem</u> 10. <u>Let</u> $f(z)$ <u>and</u> $g(z)$ <u>be in</u> A_n <u>and</u> $g(z)$ <u>is starlike for</u> $z \in D$. <u>If</u> $\mathrm{Re}\,(f'(z)/g'(z)) > 0$ <u>in</u> D, <u>then</u> $f(z)$ <u>is convex and univalent for</u> $|z| < r_n'$, <u>where</u> $r_n' = ((2n+1) - \sqrt{(2n+1)^2 - 1})^{1/n}$.

<u>This result is sharp.</u>

<u>Proof.</u> Let $f'(z)/g'(z) = h(z)$, where $h(z)$ satisfies the conditions of lemma 1.

Thus

$$f'(z) = h(z)\, g'(z)$$

The logarithmic diff. of $f(z)$ yields

$$\frac{f''(z)}{f'(z)} = \frac{h'(z)}{h(z)} + \frac{g''(z)}{g'(z)}$$

or

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{z\,h'(z)}{h(z)} + \frac{zg''(z)}{g'(z)} \tag{5.5.3}$$

As $g(z)$ is starlike in D therefore

$$\frac{z\,g'(z)}{g(z)} = \frac{1 - w(z)}{1 + w(z)}$$

follows from lemma 2.

Thus

$$1 + \frac{z\, g''(z)}{g(z)} = \frac{2z\, w'(z)}{1-(w(z))^2} + \frac{1 - w(z)}{1 + w(z)}$$

Therefore

$$(5.5.4) \qquad \mathrm{Re}\, (1 + \frac{zg''(z)}{g'(z)}) \geq \frac{1 - 2(n+1)\ |z|^n + |z|^{2n}}{1 - |z|^{2n}}$$

The last result has been obtained by using the lemma 4 and the fact that

$$\mathrm{Re}\ (\frac{1 - w(z)}{1 + w(z)}) = \frac{1 - |w(z)|^2}{|1+ w(z)|^2} .$$

Using (5.5.4) and lemma 1 in (5.5.3) we obtain

$$\mathrm{Re}\ (1 + \frac{zf''(z)}{f'(z)}) \geq \frac{-2n|z|^n}{1-|z|^{2n}} + \frac{1 - 2(n+1)|z|^n + |z|^{2n}}{1 - |z|^{2n}}$$

$$= \frac{1 - 2(2n+1)\ |z|^n + |z|^{2n}}{1 - |z|^{2n}}$$

Therefore for $|z| < r_n'$, where r_n' is same as stated in the theorem, $f(z)$ is convex and univalent. To show that the result is sharp, consider

$$f(z) = \int_0^z \frac{(1+s^n)}{(1-s^n)^{(2+2n)/n}} \quad \text{and} \quad g(z) = \frac{z}{(1-z^n)^{2/n}}$$

It is easy to check that conditions of the theorem are satisfies, but

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 + 2(2n+1)\, z^n + z^{2n}}{1 - z^{2n}} = 0 .$$

at $z = -r'_n$.

Therefore $f(z)$ ceases to be convex and univalent in any circle $|z| < r$, if $r > r'_n$.

<u>Theorem</u> 11. <u>Let</u> $f(z)$ <u>be in</u> A_n <u>and</u> $g(z)$ <u>be in</u> $C_n(\alpha)$. <u>If</u> $\operatorname{Re}\,(f'(z)/g'(z)) > 0$ <u>in</u> D, <u>then</u> $f(z)$ <u>is convex and univalent for</u> $|z| < r_n(\alpha)$, <u>where</u>

$$r_n(\alpha) = \left(\frac{\sqrt{\alpha^2 - 2n\alpha + n^2 + 2n} - (n+1-\alpha)}{2\alpha - 1}\right)^{1/n}$$

if $\alpha \neq 1/2$,

$$r_n(1/2) = (1/(2n+1))^{1/n} \quad \text{if } \alpha = 1/2 .$$

<u>This result is sharp.</u>

<u>Proof.</u> From theorem 9 we have

$$f'(z) = h(z)\, g'(z)$$

and

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zh'(z)}{h(z)} + \frac{z\, g''(z)}{g'(z)} \tag{5.5.5}$$

Since $g(z) \in C_n(\alpha)$ therefore

$$1 + \frac{zg''(z)}{g'(z)} = \frac{1 + (2\alpha - 1)\, w(z)}{1 + w(z)}$$

Thus

$$\text{(5.5.6)} \qquad \operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) \geq \frac{1 + (2\alpha-1)|z|^n}{1 + |z|^n}$$

follows from lemma 3.

Using (5.5.6) and lemma 1 in (5.5.5), we obtain

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \frac{1 + (2\alpha-1)|z|^n}{1 + |z|^n} - \frac{2n\,|z|^n}{1-|z|^{2n}}$$

$$= \frac{1 - (2n - 2\alpha + 2)|z|^n - (2\alpha-1)|z|^{2n}}{1 - |z|^{2n}}$$

Therefore for $|z| < r_n(\alpha)$, where $r_n(\alpha)$ is as stated in the theorem, $f(z)$ is convex and univalent. To show that the result is sharp, consider

$$f(z) = \int_0^z \frac{1 + s^n}{(1 - s^n)^{(n+2-2\alpha)/n}}\, ds, \quad g(z) = \int_0^z \frac{1}{(1-s^n)^{(2-2\alpha)/n}} \cdot ds$$

It is easy to check that conditions of the theorem are satisfied, but

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 + (2n - 2\alpha + 2)\, z^n - (2\alpha - 1)\, z^{2n}}{1 - z^{2n}} = 0$$

at $z^n = -r_n(\alpha)$.

Therefore $f(z)$ can not be convex and univalent in any circle $|z| < r$, if $r > r_n(\alpha)$.

<u>Theorem</u> 12. <u>Let</u> $f(z) = z + \sum_{k=n}^{\infty} a_{k+1} z^{k+1}$, $n \geq 1$ <u>be starlike in</u> D. <u>If</u> $|zf'(z)/f(z) - 1| < 1$, <u>then</u> $f(z)$ <u>is convex and univalent for</u> $|z| < r''_n$, <u>where</u>

$$r''_n = \left(\frac{(2+n) - \sqrt{n^2+n}}{2} \right)^{1/n} .$$

<u>This result is sharp.</u>

<u>Proof.</u> Let $(z f'(z)/f(z) - 1) = h(z)$, where $h(z)$ is regular and $|h(z)| < 1$ in D, and $h(0) = 0$. Therefore $h(z) = z^n \phi(z)$ where $\phi(z)$ is regular in D. Schwarz's lemma enable us to state that $|\phi(z)| \leq 1$ in D. Therefore

$$(5.5.7) \qquad \frac{zf'(z)}{f(z)} = 1 + z^n \phi(z) = 1 + w(z)$$

The logarithmic diff. of (5.5.7) yields

$$(5.5.8) \qquad 1 + \frac{zf''(z)}{f'(z)} = 1 + w(z) + \frac{zw'(z)}{1+w(z)}$$

Therefore

$$(5.5.9) \qquad \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq 1 - |z|^n - \frac{n|z|^n}{1-|z|^n}$$

$$= \frac{1 - (n+2)|z|^n + |z|^{2n}}{1 - |z|^n}$$

The inequality (5.5.9) has been obtained by using lemma 4. Therefore for $|z| < r''_n$, where r''_n is as stated in the theorem, $f(z)$ is convex and univalent. To show that the result is sharp, consider

$f(z) = z\, e^{(z^n/n)}$. It is easy to check that conditions of the theorem are satisfied, but

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 + (n+2)\, z^n + z^{2n}}{1 + z^n} = 0$$

at $z^n = - r''_n$.

Thus $f(z)$ fails to be convex and univalent in any circle $|z| < r$, if $r > r''_n$.

<u>Note</u>: Results of this section are to apear [2].

REFERENCES

[1] P.L. Bajpai and Prem Singh; The radius of starlikeness of certain analytic functions; Proc. Amer. Math. Soc. (to appear).

[2] ______________; The radius of convexity of certain analytic functions in the unit disc; Indian J. Pure Appl. Math. (to appear).

[3] S.K. Bajpai and R.S.L. Srivastava; On the radius of convexity and starlikeness of univalent functions; Proc. Amer. Math. Soc., 32(1) (1972), 153-160.

[4] S.D. Bernadi; The radius of univalence of certain analytic functions; Proc. Amer. Math. Soc.; 24 (1970), 312-318.

[5] L. Bieberbach ; Über die Koeffizienten derjenigen Potenzreihen; Welche eine schlichte Abbildung des Einheit skreises vermitteln, S-B. Preuss Akad. Wiss; 138 (1916), 940-955.

[6] E.G. Calys ; The radius of univalence and starlikeness of some classes of regular functions;Compositio Math. 23 (4) (1971), 467-470.

[7] W.M. Causey and E.P. Merkes ; Radius of starlikeness of certain analytic functions; J. Math. Anal. Appl; 31 (1970), 579-586.

[8] P.J. Eenigenberg ; A class of starlike mappings of the unit disc;Compositio Math. 24 (2) (1972), 235-238.

[9] Faber ; Neuer Beweis Koebe-Bieberbachschen stzes uber konforme abbildung, K.B. Akad. Wiss. Munchen, Sitzungsberichte der math. phys. Kl. (1916), 39-42.

[10] P.R. Garabedian and M. Schiffer ; A proof of Bieberbach conjecture for the fourth coefficient; J. Rational Mech. Anal. 4 (1955), 427-465.

[11] G.M. Golusin ; Geometrische Funktionen theorie Berlin : Deustscher Verlagder Wissenchanjten (1957).

[12] T.H. Gronwall ; Sur la deformation dans la representation conferme ; C.K. Acad. Sci., Paris, 162 (1916), 249-252.

[13] R.S. Gupta ; Radius of univalence and starlikeness of class of analytic functions ; J. Aust. Math. Soc., XIV (1) (1972), 1-8.

[14] W.K. Hayman ; Multivalent functions ; Cambridge Univ. Press, London, (1958).

[15] J. Kaczmarski ; On the coefficients of some classes of starlike functions ; Bull. Acad. Polon. Des Sci. Series des Sci. Math. Astro. et. Phys. 17 (1968).

[16] W. Kaplan ; Close to convex schlicht functions ; Michigan Math. J. 1 (1952), 169-185.

[17] P. Koebe ; Über die uniformisierung beliebiger analytischer Kurven, Nachr, Ges. Wiss. Gottingen ; 1907, 119-210.

[18] R.J. Libera ; Some radius of convexity problems ; Duke Math. J., 31 (1964), 143-158.

[19] ___________ ; Meromorphic close-to-convex functions ; Duke Math. J.; 32 (1965) 121-128.

[20] ___________ ; Univalent α-spiral functions; Canadian J. Math.; 19 (1967), 449-456.

[21] R.J. Libera and A.E. Livingston ; On the univalence of some classes of regular functions ; Proc. Amer. Math. Soc., 30(2) (1971), 327-336.

[22] A.E. Livingston ; On the radius of univalence of certain analytic functions ; Proc. Amer. Math. Soc.; 17 (1966), 352-357.

[23] K. Löwner ; Untersuchungen über schlichte knoforme Abbildungen des Einheitskreises, I, Math. Ann. 89 (1923), 103-121.

[24] T.H. MacGregor ; The radius of univalence of certain analytic functions ; Proc. Amer. Math. Soc., 14 (1963), 214-220.

[25] Z. Nehari ; Conformal mapping ; MacGraw Hill Book Company Inc., (1952).

[26] K.S. Padmanabhan ; On the radius of univalence of certain analytic functions ; J. London Math. Soc; 1 (2) (1969), 226-231.

[27] ________________ ; On certain classes of meromorphic functions in the unit circle ; Math. Z., 89 (1965), 98-107.

[28] R. Pederson ; A proof of Bieberbach conjecture for the sixth coefficient ; Arch. Rational Mech. Anal. 31 (1968), 331-351.

[29] R. Pederson and M. Schiffer ; A proof of Bieberbach conjecture for the fifth coefficient ; Arch. Rational Mech. Anal ; 45 (3) (1972), 161-193.

[30] J.S. Ratti ; The radius of convexity of certain analytic functions ; Indian J. Pure Appl. Math. ; 1 (1970), 30-36.

[31] ________ ; The radius of univalence of certain analytic functions ; Math. Z ; 107 (1968), 241-248.

[32] Ram Singh ; On a class of starlike functions ; Compositio Math ; 19 (1968), 78-82.

[33] ________ ; Correction on a class of starlike functions ; Compositio Math. ; 21 (1969), 230-231.

[34] V. Singh and R.M. Goel ; On radii of convexity and starlikeness of some classes of functions ; J. Math. Soc. Japan ; 23 (1971), 323-339.

[35] L. Špaček ; Prispevek kteorii funcki prostych, casopis Pest. Mat. a Fys ; 62 (1932), 12-19.

[36] David E. Tepper ; On the radius of convexity and the boundary distortion of schlicht functions ; Thesis ; Submitted to Temple university on Oct. 11, (1968).

[37] D.J. Wright ; On a class of starlike functions ; Compositio Math ; 21 (1969), 122-124.

[38] M.R. Ziegler ; The radius of starlikeness of certain analytic functions ; Math. Z., 122 (1971), 351-354.

[39] V.A. Zmorovič ; On the radius of convexity for starlike functions of order α in the unit circle $|z| < 1$ and in $0 < |z| < 1$; Mat Sb. (N.S.) 68 (10) (1965), 65-70.

[40] ________ ; On the bounds of starlikeness and univalence in certain classes of functions regular in the circle $|z| < 1$ Ukrain Mat. Ž ; 18(3) (1966), 28-39.

[41] Susheel Chandra ; A study of certain univalent and Multivalent analytic functions; Thesis ; submitted to the Indian Institute of Technology Kanpur, June 1973.

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